Mathematical error patterns observed with first-year university students

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Abstract

Mathematical error pattern analysis is the study of errors in learners’ work with a view to finding possible explanations for these errors. This paper provides an analysis of common mathematical error patterns observed from first-year university nmathematics students. The analysis provides some insight into the more common procedural and conceptual errors evident in the learners’ calculations. It also discusses possible remedial measures to address specific error patterns and argues that error analysis is an integral part of mathematics teaching and learning.

Keywords: Mathematical formulas, mathematical error, pattern, procedural, conceptual, arithmetic.

Introduction

Mathematics is a study of patterns, quantities, structures, space and change in such diverse fields as business, health, engineering, and computer science and involves analysis, calculations, and predictions. The complexity of calculations varies depending on the nature of a mathematical problem. The mathematical concepts, procedures, and operations provide the direction during the course of problem-solving, however, the possibilities of performing incorrect calculation and conclusion cannot be neglected. The likelihood of performing an erroneous computation may depend on factors such as lack of knowledge, misunderstanding of the questions, laziness, or simply fear of the subject assuming mathematics is an area of great difficulty. Nevertheless, if students can distinguish incorrect from correct procedures, then the common errors can be avoided and the motivation for learning mathematics may improve.

This paper seeks to discuss common mathematical error patterns observed with some PNG first-year university mathematics students . The paper introduces common mathematical errors observed and the error analysis to explain the cause of the errors. The paper also seeks to provide remedial methods to highlight the distinction between correct and incorrect procedures thereby assisting students to deepen their understanding of mathematical concepts, and improve their reasoning skills.

Mathematical prerequisites

A thorough understanding of mathematics builds upon prior mathematical knowledge from the time of early school years to university level where currently used concepts build upon prior concepts. Furthermore, concepts from different math branches interconnect with each other and misunderstanding one concept could lead to a chain of misconceptions that can later result in students having difficulty in understanding mathematics as a whole. Thus, it is imperative that students grasp the prerequisite concepts together with an understanding of proper procedures that follows within a particular area of mathematics.

> "A seemingly small gap in comprehension or knowledge creates further misapprehensions that are built one upon another, and which after some time are revealed in an error avalanche. An unrevealed error, which is rooted in the mind of students, is, therefore, a major threat to the construction of students' mathematical knowledge. A revealed and clarified error may be extremely useful for both students and teachers”. (Krygowska & Booker, 1988 pg.141)

Thus, having a broad understanding of concepts reduces chances of making erroneous computations. Even if mathematics is taught on a topic by topic basis, it is beneficial to keep an open mind on other mathematical areas in order to seek and solve problems efficiently.

Mathematical errors

Mathematical errors occur when the computation of a problem yields an unintended outcome. The problem may have been impossible to solve in the first place or some type of mathematical error may have been made during the calculation. The errors can be basic arithmetic mistakes, incorrect procedures used or conceptual misunderstanding For example, we consider one of the common errors that result from applying an incorrect procedure.

Student A: $(a+b)^2 = a^2 + b^2$

The procedure of student 1 is to distribute the exponent inside the parenthesis, which is allowable as long as there are no operators within the parenthesis. The mathematical equivalent of $(a+b)^2 = (a+b)(a+b)$ which is the correct procedure for the computation is shown in student 2's calculation.

Student B: $(a+b)^2 = (a+b)(a+b) = a^2 + ab + b^2$

Although it is crucial to understand correct procedures for any given problem, there are some special cases were general procedures may not hold. Thus, it is equally important to understand incorrect procedures to avoid mathematical errors.

Shown below (Table 1) are the results of a pre-course test given to thirty-six freshman mathematics students at Divine Word University covering wide range of mathematical topics. The investigation categorized these common mathematical errors into three main categories: arithmetic errors, procedural errors and conceptual errors.

Table 1 Error statistics assembled from a freshman pre-course test.

Topics	Arithmetic Errors	Procedural Errors	Conceptual Errors	Total Errors

Algebra	7	11	10	28
Statistics	9	10	12	31
Calculus	10	14	19	41
Trigonometry	4	8	10	22
Geometry	2	5	3	10
Probability	5	10	11	26

The table suggests that the identified error types are common in all topics. A Student may be familiar in one area , but may lack some skill in other areas. The discussion suggests teaching correct procedures is not enough, teaching how to avoid errors is also a crucial part of mathematics learning.

Computational or arithmetic errors

Computation is an integral part of mathematics ranging from simple arithmetic addition to evaluating a hyper-graph in a food web, or even more complex calculations such as computing the intensity of a black hole in a distant galaxy. The validity of the computation depends on the correct solution or the outcome. However, some computations are long and tedious, and the chances of errors occurring are high. The computational or arithmetics errors occur mostly with mathematical operations when doing addition, subtraction, division, multiplication, and indices or even in sets, logic and other operators. This type of error could result from student carelessness or lack of attention to the calculation.

Example 1

When simplifying a lengthy expression such as shown in Table 2, four students were given the same problem to compute and all four had a different set of solutions. Which student computed the correct solution is of interest for our discussion. The problem is to evaluate the expression:: -3-9 + 2 x 3+6/2-(3+1).

Here (Table 2) we have solutions from each of students A, B, and C, who each applied a different order of operation to compute the solution.

Table 2 Computations of three different students.

Student A.	Student B.	Student C
3-9 + 2 x 3+6/2-(3+1) = -10 x 2 = -20	-3-9 + 2 x 3+6/2-(3+1 = -11 x 4 = -44	-3-9 + 2 x 3+6/2-(3+1) -30 x -1 = 30

All the computations resulted in incorrect solutions. There are multiple incorrect solutions because each student applied the required operators in a different order of precedence, each being incorrect . To compute the correct result, students must consider the order of operation. Thus, it was evident in this exercise make that simple mistakes are made by not paying attention to signs. Since the expression consists of more than one mathematical operator, the BODMAS (Brackets, Of, Division, Multiplication, Addition, and Subtraction) rule must be used to identify the correct order of precedence.

However, student D used the correct order of operations to deduce the correct solution, as shown here.

Student D. 9 + 2 x 3+6/2-(3+1) = -11 + 6+2-4 = -8

Example 2

The other common error that was observed when working with indices, especially with a parenthesized expression such as $(5x)^2$. The solution stated in some responses was

$$(5x)^2 = 5x^2,$$

which is deemed incorrect with respect to indices laws. When dealing with exponents, it is important to recall that only the quantity immediately to the left of the exponent is subject to the exponent. However, in this case, parentheses are on the immediate left of the exponent, which implies that the exponent must be applied to the entire expression within the parenthesis . Thus, the correct solution would be

$$(5x)^2 = 25x^2.$$

These are a few examples to demonstrate that the errors mainly occur when there is carelessness or lack of attention to the computation during the calculation process

Procedural errors

Mathematics is driven by procedures or algorithms which are simply the steps to be followed or the methods involved in arriving at solutions. Although there may be many procedures available, each problem requires a correct algorithm to deduce an intended outcome or a solution, but the investigation revealed that students often use the incorrect procedure or in some cases incorrectly apply the algorithm. This pattern of errors is called a procedural error. It is also worth noting that an incorrectly applied algorithm may accidentally produce a correct result, which may further affirm one's misconception with certain procedures (. See Figure 1).

Some students tend to follow through given examples rather than understanding how or why the procedure works. This sets them on a path to make more procedural errors. Here we analyze some common procedural errors found to occur in first-year mathematics classes.

Error patterns in discrete mathematics

Discrete mathematics involves areas such as logic, sets, combinatorics, number theory, graph theory and other branches in mathematics as well. However, this research investigated special examples where the probability of making a procedural error is most likely. A simple example to demonstrate this procedural error is (Figure 1) where the problem is to prove that two different positive numbers are actually equal. In order to prove such as this (fallacious) conjecture, each step has to be correct and the logic must flow correctly. A mistake made in any step could result in an incorrect conclusion as illustrated in figure 1.

Student E
Problem: Proving two different positive integers numbers "a" and "b" are equal

Procedure/Steps	Reason
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $(a - b)$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b
Contradiction!	

Figure 1. Student A proving the existence of two different positive integers are equal.

The evaluation suggests no two different positive integers can be equal to each other in an integer domain. Every step in **student E** work is valid except step (5), where the student divides both sides by $(a - b)$, but by assumption $a - b$ equals zero and a division of both sides of an equation by the same quantity is valid only if the quantity is not equal to zero. However, the conclusion claims there are no existence of two different positive numbers are equal to each other in integer domain. The conclusion is true since no two different integers can equal to each other but the procedure is fallacious. The example demonstrates the possibility of errors that occur due to fallacious logic though the conclusion can be true.

The other common error pattern observed deals with negated compound statements in mathematical logic. When evaluating such statements, we use a special procedure called De Morgan's law, which states that $\neg(p \wedge q) = \neg p \vee \neg q$ and $\neg(p \vee q) = \neg p \wedge \neg q$. The problem was given to evaluate the logical equivalence of the compound statement $\neg(p \vee q \wedge r)$, p equals John is over 18 years old, q equals lives away from home and r equals has his own apartment,

Student F states that $\neg(p \vee q \wedge r)$is logically equivalent to $\neg p \vee \neg q \wedge \neg r$. The translation would mean, it is not true that John is over 18 years old or lives away from home and has his own apartment, then it is true that he is not over 18 years old or he does not live away from home and does not have his own apartment.

However, using De Morgan's law the correct equivalent to the compound statement $\neg(p \vee q \wedge r)$ is $\neg p \wedge \neg q \vee \neg r$. This implies that it is not true that John is over 18 years old and does not live away from home or has his own apartment, then it is true that he is not over 18 years old or he does not live away from home and does not have his own apartment. The latter conclusion makes more sense than the former.

The correct procedure is to understand that the complement of the conjunction of two logical statements is equal to the disjunction of their complements and complement of the two disjunctions is equals to the conjunction of their complements.

Error pattern in calculus

The common error observed in this area of mathematics is also interesting. Calculus has different techniques or procedures to follow when solving problems. While the techniques generalize, there are special cases where caution is required. This example explores the power rule procedure in differential calculus in its generality and points a special case where power rule procedure is often misused. The power rule for differentiation states that $f(x) = x^n + C$ is equal to $f(x) = (n-1)x^{(n-1)} + 0$. Example:
$f(x) = x^3 + x^2 + 5$is equals to $f'(x) = 3x^2 + 2x + 0$

However, there is a special case where the power rule is applied incorrectly. For example, the function $f(x) = x^x$ was given to students to evaluate its gradient function and the result is as shown below.

Student G

The derivative of a function with a dynamic exponent $f(x) = x^x$ was computed incorrectly using power rule as follows.

$$\boxed{f'(x) = x^x = x.x^{x-1} = x^x}.$$

This computation concludes that the derivative or the gradient function of $f(x) = x^x$ is the same function $f(x) = x^x$, as is the case of a derivative of e^x is e^x. The conclusion is incorrect even though it might appear to follow the power rule procedure. However, the function is exponentiated with a dynamic variable and not a constant The correct procedure to use in this case would be implicit differentiation (figure 2).

$y = f(x) = x^x$	
$y = x^x$	
$\ln y = \ln(x^x)$	Take the natural log (ln) on both sides
$\ln y = x \ln x^x$	By rule of Natural log.
$\frac{dy}{dx}(\ln y) = \frac{dy}{dx}(x \ln x)$	Taking derivative on both sides
$\frac{dy}{dx} \cdot \frac{1}{y} = 1 \cdot \ln x + \frac{1}{x} \cdot x$	By chain and product rule
$\frac{dy}{dx} = y(\ln x + 1)$	Multiplying y on both sides
$\frac{dy}{dx} = x^x \ln x + 1$	Substitute y

Figure 2 Correct procedure for finding the derivative of x^x

Student H

There is a similar case in the power rule for integration where it works well in its generality but fails in a special case. For the integral power rule, the set of all antiderivatives of a function *f(x)* is the indefinite integral of *f* with respect to *x* and is denoted by:

$$\int \left(x^n\right) dx = \frac{x^{n+1}}{n+1} + C .$$

Consider the example where $f(x) = x^2 + 2$ is equal to $\mathrm{f(x)} = \frac{\mathrm{x}^3}{3} + 2\mathrm{x} + \mathrm{c}$. The integral power rule holds in the integer domain except for n = -1, that is for $\int \mathrm{x}^{-1}\mathrm{dx}$. This is a special case where other methods are required. The likely erroneous calculation is as follows.

$$\boxed{\int \left(\frac{1}{x}\right) dx = \int \left(x^{-1}\right) dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} \ \textit{Solution cannot be determined}}$$

Figure 3 Student D incorrect procedure

Thus, one may conclude that the solution to the problem can not be determined. However, the solution to the given problem is shown here, which is one of the standard integration.

$$\int x^{-1} dx = \int (1/x) = \ln x + C .$$

Furthermore, understanding this procedure tends to challenge student's earlier understanding of the power rule. Student D could resolve to compute any negative power functions in the same way as function $\int x^{-1} dx$. For example, by adopting the idea from the procedure stated earlier, the function *f(x) =1/x²* can be evaluated as $\int x^{-2} dx = \int (1/x^2)\, dx = \ln(x^2) + C$. This, however, is not the correct procedure or the solution. The result is not correct since the solution can be determined with the power rule as shown below:

$$\boxed{\int (1/x^2)\, dx = \int x^{-2} dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x} + C}$$

Figure 4. Correct Solution

Procedural knowledge involves skills that a systematic to perform any particular task. Students attained the skills and procedures from concepts learned earlier. However, they sometimes learn procedures without adequately connecting the mathematical ideas that it relates. Thus, this could lead students to make a procedural error.

Conceptual error

National Council of Teachers of Mathematics (NCTM) in article title Principles and Standards for School of Mathematics (NCTM 2000, 3) stresses the importance of conceptual learning. Conceptual learning in mathematics always focuses on ideas and on generalizations that make connections among ideas. Nevertheless, failure to identify the connections within ideas or concepts could lead to erroneous solutions or turn problems into unsolvable.

> "Students' lack of knowledge could be a major reason why they cannot solve certain problems consistently "(Hudson & Miller, 2006)

The failure of identifying concepts or correctly link them is categorized as a Conceptual error. This type of error is common among students in first-year Mathematics and is probably due to the lack of knowledge to identify the relationships within different mathematical concepts. These lead students to conclude that the problem may be unsolvable or could lead students' computations with illogical mathematical calculations. However, some problems from one area can be expressed into another to compute its solutions. The example here emphasized the instances where integral technique is exhausted and the problem is converted to the other area of math to evaluate its solution.

Example 3.

Students are required to integrate the functions $\int \sqrt{1+x^2}\, dx$, however, with the known integral techniques the problem will be difficult or if not impossible to solve. The common errors observed are students attempting to apply known integral methods with no avail. The problem, however, is easily solved using trigonometric identities.

Since this problem closely resembles Pythagorean identity, the ideas in trigonometry are used as shown.

Here we convert $\int \sqrt{1+x^2}\, dx$ to trigonometric integral using right angle triangle and the Pythagoras theorem.

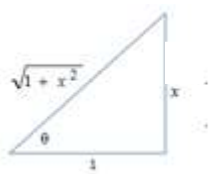

Figure 5(a)

Hence, we can reformulate the equation as $\tan\theta = x/1$ and substitute the details back to the original function (figure 5(b)).

$$x = \tan\theta$$
$$dx = \sec^2\theta\, d\theta$$
$$\begin{aligned}\int \sqrt{1+\tan^2\theta}\cdot\sec^2\theta\, d\theta \;\Rightarrow\; &\int \sqrt{\sec^2\theta}\cdot\sec^2\theta\, d\theta \\ &= \int \sec\theta\cdot\sec^2\theta\, d\theta \\ &= \int \sec^3\theta\cdot\, d\theta \\ &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln(\sec\theta + \tan\theta) + C\end{aligned}$$

Figure 5(b). Problem expressed in terms of trigonometry.

Furthermore, we translate the problem back into its original form using the triangle again.

$$\tan\theta = x$$
$$\sec\theta = \frac{\sqrt{1+x^2}}{1}$$

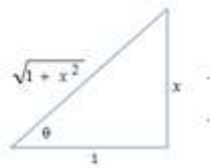

Figure 5(c)

Finally, we substituting $\tan\theta$ with x and $\sec\theta$ with $\sqrt{1+x^2}$, and the solutions obtained (figure 5d).

$$\boxed{\begin{aligned}\int\sqrt{1+x^2}\,dx &= \frac{1}{2}\sec\theta\tan\ \theta+\frac{1}{2}\ln(\ \sec\theta+\tan\ \theta\theta+C\\ &=\frac{1}{2}x\sqrt{1+x^2}+\frac{1}{2}\ln(\sqrt{1+x^2}+x+C\end{aligned}}$$

Figure 5(d) Final solution

Example 4.

The failure to identify concepts is also discussed with another problem involves limits. Students evaluate the limit of the function state:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}.$$

To evaluate the limit, students tend to substitute 1 for x as shown here

$$\lim_{x\to 1}\frac{1^2-1}{1-1}=\frac{0}{0}=0$$

It is trivial to conclude that with zero being the denominator, such limit is undefined and so cannot be determined. However, the solution can be found using with a different approach which is to apply some preliminary algebra and factorize the numerator as the difference of squares. The common factor (x – 1) of the numerator and the denominator may then be cancelled leaving the final expression as $x+1$. Finally, when we take the limit the solution is given as shown.

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1}(x+1) = 1 + 1 = 2$$

The values of *f(x)* at *x* close to 1 were computed as shown below:

Table 4.1 Result of limit as x approaches 1.

x	0.05	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	0.7	0.7	0.8	0.8	0.9	0.9	1
f(x)	1.05	1.1	1.2	1.2	1.3	1.3	1.4	1.4	1.5	1.5	1.6	1.6	1.7	1.7	1.8	1.8	1.9	1.9	2

From the table, it is clear that as the *x* values approach 1 from both the positive and the negative sides, the *f(x)* value approaches 2. Thus, the method of applying preliminary algebra to the original function before evaluating its limit could lead to a solution.

Students can solve difficult problems easily if they understand the concept links or relationship that exists within mathematics itself. Thus, we observed that students sometimes fail to identify the relationships within sub-braches of mathematics to perform computations and thus, their calculations yield unintended results.

Error analysis model.

The study shows that teaching and learning mathematics not only requires teaching correct procedures, formulas, and concepts but also requires raising awareness of possible errors that are likely to occur. Error analysis identifies the limitations of concepts and methods. It is much more than a diagnostic tool for determining a student's procedural effectiveness; often, it provides a window for determining a student’s lack of basic conceptual understanding. It clarifies the limitations of any given concept and could deepen students’ understanding of mathematical concepts, improve their reasoning skills and perfect their computational skills. Figure 6 shows the Mathematical knowledge cycle with error analysis as an integral part of teaching and learning mathematics.

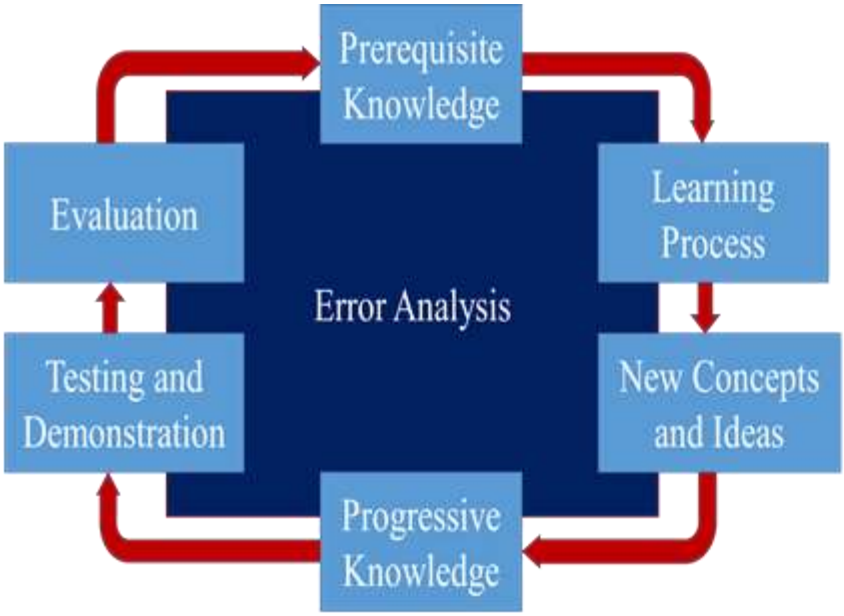

Figure 6: Mathematical knowledge cycle with error analysis as an integral part of teaching and learning mathematics.

Summary and conclusion

This paper has discussed common mathematical errors that occur with students taking early university math courses. The research identified three main categories of errors. Each category provides an insight into why an error is likely to occur or how it occurred. The paper also provided correct procedures and solutions to distinguish the difference between correct procedures from incorrect. The correct procedures aimed to motivate students to think as they engage in solving mathematical problems. Though mathematics can be challenging to the average student, the paper also proposed that all knowledge acquired through the educational stages in a student's life are important as they provide background for the students to advance their knowledge in mathematics. The passion, dedication, and interest in the subject would enhance the interest in learning mathematics, which will deepen students' understanding of mathematical concepts, improve their reasoning skills.

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