Differentiability in normed spaces: A new approach

Raunu Gebo Sarsoruo

Abstract

The notions of limit, continuity, linearity and bilinearity are very substantial in the study of the general theory of differentiability in normed spaces. These concepts are used to provide precise proofs of differentiability of some functions in normed spaces. Common properties of the derivative of a function at a particular point are identified and expounded. The paper aims to show a new approach using common abstractive ideas to develop a better understanding of differentiation. Foundational concepts from limits that relate to continuity, then to linearity and bilinearity in the form of definitions, theorems and lemmas including some of their proofs provide a better way of understanding differentiability in calculus. Another significant result explored is the differentiability and continuity of implicit functions in Banach Spaces.

Keywords: Banach space, billinear, continuous linear mapping, continuous, converge, derivative, differentiable, implicit function limit, linear, norm, normed space, sequence.

Introduction

Interestingly, prevalent philosophical concepts that indicate the existence of continuity when a function is differentiable, are proposed to be a better way of understanding differentiation for university students and lecturers of calculus classes. Differentiation is a key concept in the study of calculus as a foundation for mathematical analysis. The main notion described is the differentiability of a function at a specific point. It seems to hold true for almost all occurring results from certain definitions, lemmas and the main theorems used. Each result seems to build upon the results from another theorem or lemma.

To achieve the main idea of this paper, which is to show that differentiability of a function at a specific point implies the existence of a limit and continuity at that same point, we begin from the notion of limits. It is known that the limit of a function f as x approaches a point a $\in \mathbb{R}$ can be found simply by calculating the value of the function f at point a. The concept of continuity builds on this property where functions are said to be continuous at a. If functions are continuous at a, then there exists a tangent line at that point. Such a line introduces the existence of a linear mapping at that point. If such a linear map exits at that point a, then we say that function f is differentiable at point a and satisfies the condition of differentiability stated as,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

This paper will also examine the differentiability of implicit functions and some observations are made when an implicit function is defined over a Banach space. Implicit differentiation is one of the many different techniques of differentiation which is useful for university students studying calculus.

Limit

The concepts of linearity, continuity and differentiability evolve firstly from the notion of limits. Limits are the foundational basis for continuity, linearity and differentiability to occur in normed spaces. Hence, the existence of a limit of a function at a point underlies the study of the theory of differentiability in general. Sequences are used to achieve a better and more concise understanding of limits.

We fix real linear spaces D, E over the same field $K \in \mathbb{R}$ and open set $V \subset D$ (Baron, 2019).

Definition 1 Function $f: \mathbb{Z}^+ \to \mathbb{R}$ given by $f(a_n)$ is called a sequence whose domain is a set of positive integers n.

Example 1 Let $f(a_n) = \frac{1}{n}$. The sequence whose n^{th} term is $\frac{1}{n}$ may be written as

$$f\{\frac{1}{n}\}_{n=1}^{\infty} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\}.$$

Remark 1 Observe that by choosing n sufficiently large, we find terms that are very close to zero. On the other hand, regardless of how large n is chosen, there are terms further out in the sequence that are not close to zero.

Example 2 Let $f(a_n) = \{\frac{(1+(-1)^n)}{2}\}_{n=1}^{\infty} = \{0,1,0,1,0,1,\cdots\}$. If we chose $n = 1{,}000{,}001$, then $a_{1,000,001} = 0$, but $a_{1,000,002} = 1$, which is not close to zero. If on a measure of closeness, say within $.025$ of zero, it is clear that all terms of the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ beyond the 40^{th} term satisfy this criterion. This idea leads to build the definition of convergence.

Definition 2 Function $f: \mathbb{Z}^+ \to \mathbb{R}$ given by a sequence $f(a_n)_{n=1}^{\infty}$ $converges$ to a real number x if and only if for every $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$ we have $|a_n - x| < \varepsilon$.

Remark 2 The choice of N depends on the choice of ε. The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to zero by intuition. If this conclusion is correct, then for $\varepsilon > 0$, there exists an N such that, for $n \geq N$, $|a_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$.

Example 3 If $\varepsilon = .035$, then, for $n \geq 51, |a_n - 0| = \frac{1}{n} \leq \frac{1}{51} < .035$. Thus, for $\varepsilon = .035, N = 51$ satisfies the conditions of definition 2. Now, we show that $f\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to zero (Gaughan, 1993, P51).

- Choose $\varepsilon > 0$.
- Let N be an integer larger than $\frac{1}{\varepsilon}$.
- If $n \geq N$, then we have $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$.
- This means that if $a_n = \frac{1}{n}$ and $A = 0$, then for $n \geq N, |a_n - A| = \frac{1}{n} < \varepsilon$.
- Therefore, the sequence converges to zero.

Definition 3 A sequence $\{a_n\}_n = 1^{\infty}$ is said to be $convergent$ iff there is a real number $x \in D$ such that $\{a_n\}_{n=1}^{\infty}$ converges to x. If $\{a_n\}_{n=1}^{\infty}$ is not convergent, it is said to be $divergent$.

Remark 3 The unique number to which a sequence converges to is called the $limit$ of the sequence.

Definition 4 Let x be an element of a real linear space D. A set $V \subset D$ is called the $neighborhood$ of point x, if there exists an open set $U \subset D$ such that $x \in U \subseteq V$.

In other words, a set V is the neighbourhood of the point x, if $x \in IntV$, where $IntV$ means the interior of the set V.

Definition 5 Let P be a subset of a real linear space D. A point $c \in P$ is called an $accumulation\, point$ of P if every neighbourhood of c contains at least one point of P different from c itself (Gaughan, 1993, P64).

Example 4 Given the set $G = \{\frac{1}{n}\}$ where n is a positive integer. This set is the range of the sequence $\{\frac{1}{n}\}_n^\infty$. Example 3 shows that $\{\frac{1}{n}\}_n^\infty$ converges to zero. Thus, every neighbourhood of 0 contains infinitely many terms of the sequence; and, since all terms of the sequence are distinct (that is, if $m \neq n$, then $a_m \neq a_n$), every neighbourhood of 0 contains infinitely many points of the set G. Therefore, 0 is an accumulation point of the set G.

Definition 6 Let $f: D \to \mathbb{R}$ and x_0 be an accumulation point of D. Then, function f has a limit L at x_0 iff for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x \in D$ if

$$0 < |x - x_0| < \delta$$

then

$|f(x) - L| < \varepsilon$ as illustrated in Figure 1.

Definition 7 Let $f: D \to \mathbb{R}$ be a function defined on some open interval that contains the number b, except possibly at b itself. Then, we say that the *limit of $f(x)$ as x approaches b is L*, and we write it as

$$\lim_{x \to b} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that if

$$0 < |x - b| < \delta$$

then

$|f(x) - L| < \varepsilon$ as illustrated in Figure 1 (Gaughan, 1993, P65).

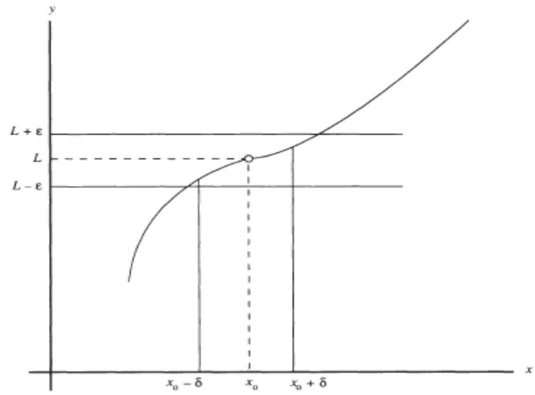

Figure 1: Illustration of $\varepsilon - \delta$ definition of limit (Gaughan, 1993, P65).

Example 5 Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ given by

$$f(x) = \frac{x^2-1}{x-1} \text{ for } x \neq 1, f(x) = \frac{x^2-1}{x-1} = x+1,$$

hence f is a linear function and the graph of f is a line with slope 1, except for $x = 1$. So, as x approaches 1, $f(x)$ approaches 2 filling in every necessary gap as possible. Hence, the limit of f at $x = 1$ is $L = 2$. Let us prove that f has a limit $L = 2$ at $x = 1$ using Figure 2.

Let us take $\varepsilon > 0$ as we consider the geometric interpretation of the idea of a limit outlined below.

- Choose a neighbourhood of 1 such that for x in this neighbourhood with $x \neq 1$.
- The corresponding points on the graph of f lie in the strip
$$\{(x, y): 2 - \varepsilon < y < 2 + \varepsilon)\}.$$
- Ignoring the point $x = 1$, the graph of f is a straight line of slope 1.
- Try $\varepsilon = \delta$ to obtain the neighbourhood $(1 - \delta, 1 + \delta)$ of $x = 1$.
- If $0 < |x - 1| < \delta = \varepsilon$, then

$$|f(x) - 2| = |\frac{x^2 - 1}{x - 1} - 2| = |(x + 1) - 2| = |x - 1| < \delta = \varepsilon$$

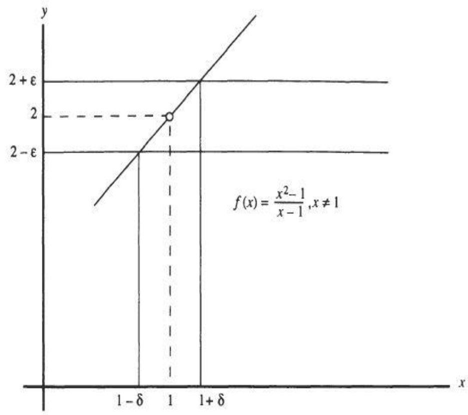

Figure 2: Example of $\varepsilon - \delta$ definition of limit (Gaughan, 1993, P66.)

Theorem 1 Let $f: D \to \mathbb{R}$ be a function with x_0 an accumulation point of D. Then, f has a limit at point x_0 if and only if for each sequence $\{x_n\}_{n=1}^{\infty}$ converging to x_0 with $x_n \in D$ and $x_n \neq x_0$ for all n, the sequence $\{f(x_n\}_{n=1}^{\infty}$ converges (Gaughan, 1993, P57).

Continuity

In the discussion of the limit of a function above, a function f has the property that if it has a limit at a point x_0, then it is said to be $continous$ at that point. Continuity in everyday language defines a process that takes place without interruption or abrupt change. Thus, a mathematical definition of continuity is closely related. Generally, we say that a function $f: D \to E$ is $continous$ at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

However, continuity can be defined more precisely using the definitions below that follows on from the notion of limits.

Definition 8 Suppose $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$. If $x_0 \in D$ then f is $continous$ at x_0 if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$|x - x_0| < \delta, x \in D$$

then

$$|f(x) - f(x_0)| < \varepsilon \text{ as displayed in Figure 3.}$$

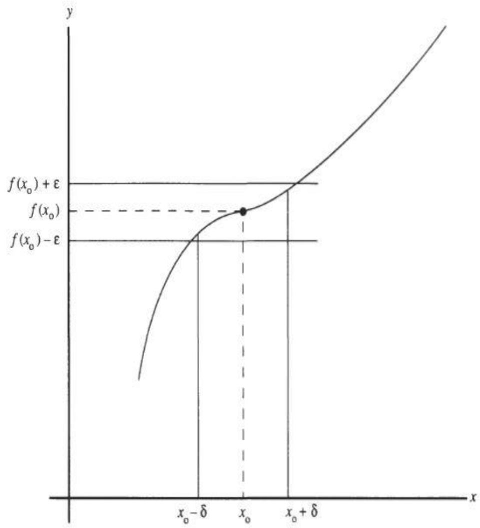

Figure 3: Illustration of $\varepsilon - \delta$ definition of continuity (Gaughan, 1993, P86.)

Remark 4 When comparing the definition (8) of continuity to the definition (6) of the limit of a function at point x_0, the following observations are made (Gaughan, 1993, P86).

- For continuity at x_0, the number x_0 must belong to D, but not an accumulation point of D.
- If $f: D \to \mathbb{R}$ with $x_0 \in D$ and x_0 not an accumulation point of D, then there is $\delta > 0$ such that if $|x - x_0| < \delta$ $x \in D$, then $x = x_0$.
- Hence $|f(x) - f(x_0)| = 0 < \varepsilon$ for every $\varepsilon > 0$.
- Now, we can say that if x_0 is not an accumulation point of D and $x_0 \in D$, then f is continuous at x_0 by default.
- Consider the case when x_0 is an accumulation point of D. Then, f has a limit at x_0 and that limit is $f(x_0)$.
- Comparing Figure 1 to Figure 3, f is continuous at x_0 if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that the graph of f for $x_0 - \delta < x < x_0 + \delta$, $x \in D$ lies in the strip $\{(x, y): f(x_0) - \varepsilon < y < f(x_0) + \varepsilon\}$.

Theorem 2 Let $f: D \to \mathbb{R}$ with $x_0 \in D$ and x_0 an accumulation point of D. Then the following conditions are equivalent (Gaughan, 1993, PP 87-88):

- For every sequence at $\{x_n\}_{n=1}^{\infty}$ converging to x_0 with $x_n \in D$ for each $n, \{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.
- f has a limit at x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$.
- f is continuous at x_0.

Proof

1. (i) $\Longrightarrow$ (ii)

   (a) Assume that (i) holds.

   (b) In particular, if $\{x_n\}_{n=1}^{\infty}$ converges to x_0 with $x_n \neq x_0$ and $x_n \in D$ for all n, then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.

   (c) Hence, by Theorem 1 f has a limit at x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$.

   (d) Thus (i) implies (ii).

2. (ii) $\Longrightarrow$ (iii).

   (a) Assume (ii) holds, and choose $\varepsilon > 0$.

   (b) Since (ii) holds, there is a $\delta > 0$ such that if $0 < |x - x_0| < \delta$ for $x \in D$, then $|f(x) - f(x_0)| < \varepsilon$.

   (c) If $0 < |x - x_0|$, then $|x - x_0| = 0$ implies that $x = x_0$.

   (d) Hence, $|f(x) - f(x_0)| = 0 < \varepsilon$.

   (e) Thus, f is continuous at x_0 and (ii) implies (iii).

3. (iii) $\Longrightarrow$ (i).

   (a) Suppose now that (iii) holds and that $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in D that converges to x_0.

   (b) Choose $\varepsilon > 0$. There is a $\delta > 0$ such that for $|x - x_0| < \delta$, for $x \in D$< $|f(x) - f(x_0)| < \varepsilon$.

   (c) Since $\{x_n\}_{n=1}^{\infty}$ converges to x_0, there is N such that for $n \geq N, |x_n - x_0| < \delta$.

   (d) Thus, for $n \geq N, |f(x_n) - f(x_0)| < \varepsilon$.

(e) This shows that $\{f(x_n)\}_{n=1}^{\infty}$. converges to $f(x_0)$.

Definition 9 Let a_i be a sequence of numbers for $a_i \in \mathbb{R}$. Sequence (a_i) is a Cauchy sequence, if

$$\underset{\varepsilon\in\mathbb{R}}{\forall} \quad \underset{\varepsilon\geq 0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{m,n>N}{\forall} \quad |a_m - a_n| < \varepsilon. \ (1)$$

This means that by selecting any small positive real number ε, a sufficiently large indicator N can be set such that any two expressions of higher orders are less than ε.

There are two equivalent definitions of real functions of a real variable. Let $M \subset \mathbb{R}$ and $f\colon M \to \mathbb{R}$.

Definition of Cauchy Function f is continuous in point $x_0 \in M$ if and only if:

$$\underset{\varepsilon\in\mathbb{R}}{\forall} \quad \underset{\delta\geq 0}{\exists} \quad \underset{x\in M}{\forall} \quad |x_0 - x| < \delta \to |f(x_0) - f(x)| < \varepsilon. \ (2)$$

The conditions

- $|x_0 - x| < \delta$ means that x belongs to the open sphere in the middle x_0 and radius δ.
- $|f(x_0) - f(x)| < \varepsilon$ means that $f(x)$ belongs to the open sphere in the middle $f(x_0)$ and radius ε.

Definition of Heine The function is continuous at point $x_0 \in M$, if and only if for each sequence (x_n) numbers from M, which is convergent to x_0, the string of values $(f(x_n))$ converges to $f(x_0)$, or

$$\underset{(x_n)\subset M}{\forall} x_n \to x_0 \to f(x_n) \to f(x_0). \qquad (2)$$

If the function f meets one of the above conditions for every $x \in M$, it is continuous on the set M respectively in the sense of Cauchy or the sense of Heine.

Linearity

We fix linear spaces W, X over the same field $K \in \{\mathbb{R}, \mathbb{C}\}$ and a set V (Baron, 2019).

Definition 10 A $linear\ (vector)\ space$ over the field K is called a set V with two binary operations defined as:

- addition of vectors: operation from the Cartesian product of the set V on set V, $+: V \times V \rightarrow V$, for vectors $u, v, w \in V$, we have $u = v + w$.
- scalar multiplication: operation from the Cartesian product of the set V and field K, $K \times K \rightarrow V$, for vectors $u, v \in V$ and number $a \in K$, we have $u = av$.

Definition 11 Let $f: W \rightarrow X$ be a mapping. We say that f is $linear$, if the following properties are satisfied;

- $f(u + v) = f(u) + f(v)$ for every elements, $u, v \in W$.
- $f(\lambda v) = \lambda f(v)$ for every $\lambda \in K$ and $v \in W$.
- $f(\lambda + \mu v) = \lambda f(u) + \mu f(v)$ for every $\lambda, \mu \in K$ and for every $u, v \in W$.
- $f(\lambda_1 v_1 + \cdots + \lambda_k v_k) = \lambda f(v_1) + \cdots + \lambda_k f(v_k)$ for every scalar $\lambda_1 \cdots \lambda_k \in K$, elements $v_1 \cdots v_k \in W$ and every $k \in \mathbb{N}$.

Property $f(u + v) = f(u) + f(v)$ is called the $additivity$ and property $f(\lambda v) = \lambda f(v)$ is called the $homogeneity$ of mapping f respectively.

Definition 12 Function $\|\cdot\|: X \rightarrow \mathbb{R}$ is called a norm in X, if for all $x, y \in X$, $\alpha \in \mathbb{K}$ the following properties hold:

- $\| x \| = 0 \rightarrow x = 0$
- $\| \lambda x \| = |\lambda| \| x \|$
- $\| x + y \| \leq \| x \| + \| y \|$

If $\|\cdot\|$ is a norm on X, then the pair $(X, \|\cdot\|)$ is called $normed\ space$.

Remark 5 If $\|\cdot\|$ is a norm in X, then for all $x, y \in X$ we have:

- $x = 0 \rightarrow \| x \| = 0$
- $\| x \| \geq 0$
- $| \| x \| - \| y| \leq \| x - y \|$ $\quad (0 = \| 0 \| = \| x + -x \| \leq \| x \| + \| -x \| = 2 \| x \|)$

Definition 13 Let X denote any non-empty set. A $metric$ on set X is called function $d: X \times X \rightarrow [0, +\infty)$ which for any elements $x, y, z \in X$ satisfy the conditions:

- $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

If d is a metric in set X, then the pair (X, d) is called $metric\ space$. Elements of set X are called $points$. Number $d(x, y)$ is called the $distance$ of a point x from point y.

Remark 6 If $\|\cdot\|$ is a norm on X, then function $d: X \times X \rightarrow [0, \infty)$ given by the formula $d(x, y) = \| x - y \|$ is a $metric$ on X.

Definition 14 Normed space $(X, \|\cdot\|)$ is called $Banach\ space$ if the metric space (X, d) is $complete$.

Remark 7 The completeness of the metric means that each $Cauchy\ sequence$ of elements in space X is convergent to some element of space X.

Differentiability in Normed Spaces

Let us fix normed spaces X, Y over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and open set $U \subset X$ (Baron, 2019).

Definition 15 Function $f: U \to Y$ is called differentiable in point $x_0 \in U$ if and only if when there exists such a continuous linear mapping $\Lambda: X \to Y$, that

$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)-\Lambda h}{\|h\|} = 0. \qquad (3)$$

Function $f: U \to Y$ is called $differentiable$ if and only if, when it is differentiable in every point of set U.

Remark 8 If function $f: U \to Y$ is differentiable in point $x_0 \in U$, then there exists exactly $one\ continous\ linear\ mapping$ $\Lambda: X \to Y$ satisfying condition (4). The continuity in this mapping means that:

$$\underset{\varepsilon>0}{\forall} \ \underset{\delta>0}{\exists} \ \underset{h \in X}{\forall} \ (0 < \| h \| < \delta \Longrightarrow \| \frac{f(x+h)-f(x)-\Lambda h}{\|h\|} \| < \varepsilon). \qquad (5)$$

- Assume that $\Lambda_1, \Lambda_2: X \to Y$ are linear mappings and $\lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0)-\Lambda_k h}{\|h\|} = 0$, for $k \in \{1,2\}$
- $\lim_{h \to 0} \frac{\Lambda_1 h - \Lambda_2 h}{\|h\|} = \lim_{h \to 0} (\frac{f(x_0+h)-f(x_0)-\Lambda_2 h}{\|h\|} - \frac{f(x_0+h)-f(x_0)-\Lambda_1 h}{\|h\|}) = 0.$
- Take $\varepsilon > 0$.
- $\delta \in (0, \infty): \forall_{h \in X} \quad (0 < \| h \| < \delta \Longrightarrow (\| \frac{\Lambda_1 h - \Lambda_2 h}{\|h\|} \| \leq \varepsilon).$
- If $h \in X\{0\}$, then $0 < \| \delta \frac{h}{\|h\|} \| = \delta$ and therefore $\| \frac{(\Lambda_1 - \Lambda_2) \frac{\delta h}{\|h\|}}{\| \frac{\delta h}{\|h\|} \|} \| \leq \varepsilon$ that is $\| (\Lambda_1 - \Lambda_2) h \| \leq \delta \| h \|$
- $(\Lambda_1 - \Lambda_2) h = 0$ for $h \in X$
- $\Lambda_1 = \Lambda_2$

Example 6 Show that if $-\infty < a < b < \infty$, then function $f: C([a,b]) \to \mathbb{R}$ defined by the formula:

$$f(x) = \int_a^b \alpha(t) x(t)^2 \ dt$$

is differentiable and

$$f'(x)h = 2\int_a^b \varphi(t)x(t)h(t)dt$$

for $x, h \in C([a,b])$.

First, we check if the derivative of function $f(x)$ satisfies the notion of linearity and continuity. We fix

$$x \in C([a,b])$$

and define the function

$$\Lambda : C([a,b]) \to \mathbb{R}$$

by the formula

$$\Lambda h = \int_a^b \varphi(t)x(t)^2 \ dt$$

- $Linearity$:

  We fix elements $h, k \in C([a,b])$ and $\alpha \in \mathbb{C}$

- $Additivity$:

$$\Lambda(h+k) = 2\int_a^b \varphi(t)x(t)h(t) + k(t)dt$$

$$= 2\int_a^b \varphi(t)x(t)h(t) + \varphi(t)x(t)k(t)dt$$

$$= 2(\int_a^b \varphi(t)x(t)h(t)dt + \int_a^b \varphi(t)x(t)k(t)dt)$$

$$= 2\int_a^b \varphi(t)x(t)h(t)dt + 2\int_a^b \varphi(t)x(t)k(t)dt$$

$$= \Lambda(h) + \Lambda(k)$$

- $Homogenity$:

$$\Lambda(\alpha h) = 2\int_a^b \varphi(t)x(t)\alpha h(t)dt =$$

$$\alpha \cdot 2\int_a^b \varphi(t)x(t)k(t)dt = \alpha\Lambda(h)$$

- $Continuity$:

  We know that, $\| h \| = sup_{t\in[a,b]}|h(t)|$ for $h \in C([a,b])$.

$$|\Lambda h| = 2|\int_a^b \varphi(t)x(t)h(t)dt| \leq$$
$$\leq 2|\int_a^b |\varphi(t)x(t)h(t)dt| =$$
$$= 2|\int_a^b |\varphi(t)|\cdot|x(t)|\cdot|h(t)|dt| \leq$$
$$\leq \| h \| \underbrace{2\int_a^b |\varphi(t)|\cdot|x(t)|dt}_{\text{constant based on h}}$$

- $Differentiability$:

$$\lim_{h\to 0}\frac{|f(x_0+h)-f(x_0)-\Lambda h|}{\|h\|} =$$
$$\frac{|\int_a^b \varphi(t)(x+h)(t)^2dt-\int_a^b \varphi(t)x(t)^2dt-2\int_a^b \varphi(t)xh(t)dt|}{\|h\|}$$
$$= \frac{|\int_a^b \varphi(t)([(x(t)+h(t))(t)]^2-x(t)^2-2xh(t))dt|}{\|h\|}$$
$$=$$
$$\frac{|\int_a^b \varphi(t)x(t)^2+2\varphi(t)x(t)h(t)+\varphi(t)h(t)^2-\varphi(t)x(t)^2-2\varphi(t)x(t)h(t))dt|}{\|h\|}$$
$$= \frac{|\int_a^b \varphi(t)(h(t)^2)dt|}{\|h\|}$$
$$\leq \frac{\int_a^b |\varphi(t)|+(|h(t)^2|\leq\|h\|^2|)dt}{\|h\|}$$
$$\leq \frac{\|h\|^2\int_a^b |\varphi(t)|dt}{\|h\|} = \| h \| \int_a^b |\varphi(t)|dt \Longrightarrow^{h\to 0} 0$$

(Baron, 2019).

Definition 16 If function $f: U \to Y$ is differentiable in point $x_0 \in U$, then one continuous linear mapping $\Lambda: X \to Y$ satisfying condition (4) is called $derivative\ of\ function\ f$ at point x_0 and is denoted by the symbol $f'(x_0)$ and so $f'(x_0)$ is a $continous\ linear\ mapping$ of space X in space Y and

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)-f'(x_0)h}{\| h \|} = 0.$$

***Remark* 9** The continuity in the mapping Λ means that;

$$\underset{\varepsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{h\in X}{\forall} \quad \left(0 < \| h \| < \delta \Longrightarrow \left\| \frac{f(x_0+h)-f(x_))-f'(x_0)h}{\|h\|} \right\| < \varepsilon\right). \qquad (4)$$

The following observations can be made:

- If $\Lambda: X \to Y$ is a continuous linear mapping, then Λ is a differentiable function and $\Lambda'(x) = \Lambda$ for $x \in X$.
- If X_1, X_2 are normed spaces and $\Lambda: X_1 \times X_2 \to Y$ are continuous billinear mapping, then Λ is differentiable and $\Lambda'(x_1, x_2)(h_1, h_2) = \Lambda(x_1, h_2) + \Lambda(h_1, x_2)$ for $(x_1, x_2), (h_1, h_2) \in X_1 \times X_2$.
- Homogeneity in each variable is bilinear such that we have $\Lambda(\alpha_1 x_1, \alpha_2 x_2) = \alpha_1 \Lambda(x_1, \alpha_2 x_2) + \alpha_2 \Lambda(x_1, x_2)$ (Baron, 2019).

Lemma 1 If X_1 and X_2 are normed spaces, then bilinear mapping $\Lambda: X_1 \times X_2 \to Y$ is continuous, if and only if, there exists such a constant $c \in (0, \infty)$, that

$$\| (x_1, x_2) \| \leq c \| x_1 \| \| x_2 \|$$

$$\text{for } (x_1, x_2) \in X_1 \times X_2.$$

Remark 10 If a function $f: U \to Y$ is differentiable in point $x_o \in U$, then function $u: U \to Y$ defined by the formula:

$$u(x) = \begin{cases} \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{\|x-x_0\|} & \text{if } x \neq x_0, \\ 0 & \text{if } x = x_0, \end{cases} \qquad (7)$$

is continuous in point x_0 (Baron, 2019).

Theorem 3 A function that is differentiable at a point is continuous at that same point.

Proof

- Assume that $f: U \to Y$ is differentiable in point $x_0 \in U$.
- Function $u: U \to Y$ defined by formula (7) is continuous in point x_0 and $f'(x) = f(x_0) + f'(x_0)(x - x_0) + \| x - x_0 \| u(x)$ for $x \in U$.
- $\lim_{x \to x_0} f(x) = f(x_0)$ (Baron, 2019).

Definition 17 Isom $(X, Y) = \{\Lambda: X \to Y | \Lambda$ is a one-to-one continuous linear mapping, $\Lambda(X) = Y$ and Λ^{-1} is a linear mapping.$\}$

Definition 18 Let $L(X, Y)$ denote all the family of linear and continuous operators in normed space X with respect to the values in normed space Y.

Definition 19 Function $f: U \to Y$ is called a C^1 class function if and only if, f is differentiable and $f': U \to L(X, Y)$ is continuous.

Remark 11 Continuity in the mapping, $f': U \to (X, Y)$ means that:

$$\underset{x_0 \in U}{\forall} \; \underset{\varepsilon > 0}{\forall} \; \underset{\delta > 0}{\exists} \; \underset{x \in U}{\forall} \; (0 < \| x - x_0 \| < \delta \Longrightarrow \| f'(x) - f'(x_0) \| < \varepsilon). \qquad (5)$$

Differentiability of implicit functions in Banach Space

We begin this section by expounding the concept of differentiability to functions that are impossible to differentiate directly. Observe that a function of the form $y = f(x)$ is simple to differentiate. However, when it is inconvenient to express functions of this form, we tend to use functions that are defined implicitly. For instance, consider the function

$$yx + y + 1 = x.$$

This function is not expressed in the form of $y = f(x)$, but it still defines y as a function of x since it can be rewritten as

$y = \frac{x-1}{x+1}$ (Anton et al, 2012, P185).

Example 7 Consider the function $f(x) = 2y + x + 8 = 1$. Applying implicit differentiation on function f with respect to x results in;

$$\frac{d}{dx}(2y) + \frac{d}{dx}(x) + \frac{d}{dx}(8) = \frac{d}{dx}(1)$$
$$2\frac{dy}{dx} + 1 + 0 = 0$$
$$\frac{dy}{dx} = \frac{-1}{2}$$

We fix Banach spaces X, Y, Z and open set $D \subset X \times Y$ (Baron, 2019).

Theorem 4 If $f: D \to Z$ is a C^1 class function, $(x_0, y_0) \in D, f'_Y(x_0, y_0) \in Isom(Y, Z)$ and $f(x_0, y_0) = 0$, then there exists a neighbourhood $U_0 \subset X$ of point x_0, a neighbourhood $W_0 \subset Y$ of point y_0 and such a function $\varphi: U_0 \to Y$ of C^1 class function, that $U_0 \times W_0 \subset D, \varphi(U_0) \subset W_0, \underset{x,y \in U_0 \times W_0}{\forall} (f(x, y) = 0 \Leftrightarrow y = \varphi(x))$

Identified from the result of theorem (4), the following observations can be made.

- For all neighbourhoods of point $x, y \in U_0 \times W_0$, some function, $f(x, y) = 0$ if and only if $y = \varphi(x)$.
- The function $y = \varphi(x)$ is a C^1 class function which is differentiable and continuous as defined in definition (19).
- If $x, y \in U_0 \times W_0$, then
$$f(x, y) = 0 \Leftrightarrow F(x, y) = (x, 0)$$
$$\Leftrightarrow F|_{U_0 \times W_0}(x, y) = (x, 0).$$

Example 8 If X, Y are Banach spaces, $D \subset X \times Y$ is an open set, $(x_0, 0) \in D$, and $g: (x_0) = 0$, then for every available number close/near 0, denoted by α, there exist a neighbourhood $U \in X$ of point x_0 and such a function $\varphi: U \to Y$ of class c^m, that $\varphi(x_0) = 0$ and $(x, \varphi(x)) \in D$ and $\varphi(x) = \alpha g(x, \varphi(x))$ for $x \in U$ (Baron, 2019).

Theorem 5 We assume that $f: D \to Z$ is a C^1 class function and $\forall_{x,y \in D}(f'_Y(x, y) \in Isom(Y, Z))$. If $U \in X$ is an open set and $\varphi: U \to Y$ is such a continuous function that

$$\underset{x \in U}{\forall} x, \varphi(x)) \in D \wedge f(x, \varphi(x)) = 0),$$

then φ is a C^1 class function and

$$\varphi'(x) = -f'_y(x, \varphi(x)^{-1} \circ f'_x(x, \varphi(x))$$

for $x \in U$.

Example 9 If $U \in \mathbb{R}^N$ is an open set, $a \in U, f: U \to \mathbb{R}$ is a C^1 class function, $f(a) = 0$ and $f'(a) \neq 0$, then set

$$\{x \in \mathbb{R}^N: \sum_{j=1}^{N} f|_j(a)(x_j - a_j) = 0\}$$

is a tangent plane to set $f^{-1}(\{0\})$ in point a (Baron, 2019).

Conclusion

Limit and continuity indeed provide a very fundamental approach to understanding differentiability in mathematical spaces. It is evident from the results of the definitions and theorems that differentiability of a function in a normed space heavily relies on the existence of a limit and continuity at a specific point. Hence, a function that is differentiable at a particular point implies that the function is continuous and has a limit at that same point. We also observe that a C^1 class function defined implicitly over Banach spaces implies differentiability and continuity of some neighbourhood points.

Glossary

C^1- A function that is differentiable and continuous.
$\mathbb{R}$ - A space of real numbers.
$\mathbb{N}$ - Natural numbers.
$L(X, Y)$- Family of linear and continuous operators.
f- a function.
f' - a differentiable function.
$f: X \rightarrow Y$- a mapping.
$V \subset D$ - an open subset.

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Author

Raunu Gebo Sarsoruo is a lecturer in the Department of Mathematics and Computing Science at DWU. Email: rgebo@dwu.ac.pg