Variation of the value of Pi on non-Euclidean surfaces

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Abstract

Pi, the ratio of circumference to diameter of a circle, is an infinite non-repeating decimal number calculated, so far, to a trillion decimal places. This ratio was considered to be a universal constant until new geometries of curved surfaces were developed in the 19^{th} c. with the famous geometry of flat surfaces developed by Euclid $(4^{\text{th}}$ c. BC) being but one example. In these non-Euclidean geometries, shortest paths are not straight lines, but great circles for spherical surfaces with positive curvature and hyperbolae for surfaces with negative curvature. This paper shows that on spheres, Pi becomes smaller as circle circumferences grow larger with the reverse occurring on hyperbolic surfaces. This topic is of general interest, given the worldwide celebration of Pi day on $3/14/$xx each year when we try to interest the general population, including students in Madang schools, colleges and universities in mathematics.

Keywords: circumference, diameter, geometry, hyperbolic, non-Euclidean, Pi.

Introduction

Interest in Pi follows renewed focus on International Day of Mathematics Day (IDM) celebrated annually on March 14 as a worldwide celebration when all countries are "invited to participate through activities for both students and the general public in schools, museums, libraries and other spaces" (IDM, 2020, para. 1).

This paper will attempt to assemble already developed accessible algebraic proofs to demonstrate that the values of Pi decrease with increasing radius for circles on the surface of a sphere and increase on hyperbolic surfaces. We approach this both by considering the ratio of circle circumference to the radius on each surface type and also by considering the ratio of circle area to the radius on each surface (IDM, 2020).

Preliminary

We note that ordinary trigonometric functions $\sin\theta$, $\cos\theta$, and $\tan\theta$ form points on a unit circle. In hyperbolic geometry, we have corresponding functions $\sinh(x), \cosh(x)$ and $\tanh(x)$, which represent points on the right half of an equilateral hyperbola, hyperbolic sine and hyperbolic cosine. Whilst θ refers to angles subtended at the centre of a circle, x refers to positions on the x axis (Figure 1).

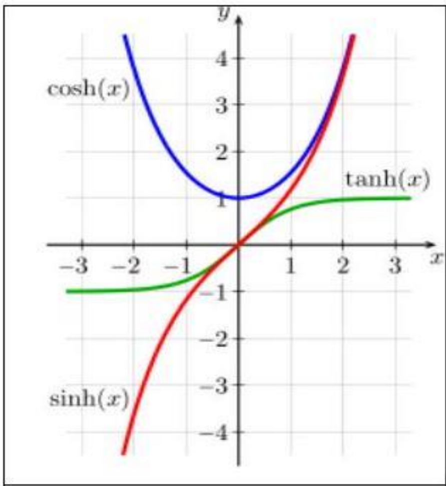

Figure 1
Source:
https://brilliant.org/wiki/hyperbolic-trigonometric-functions/

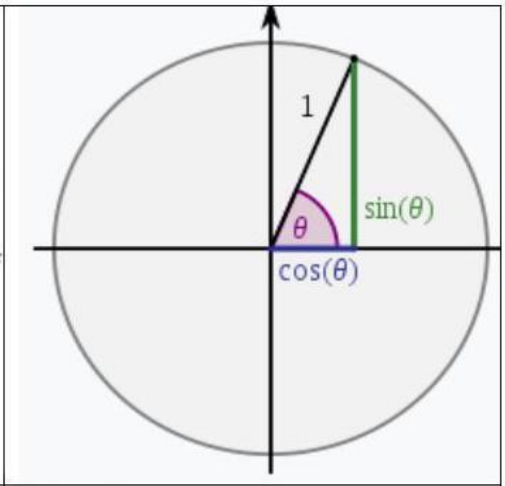

Figure 2
https://en.wikipedia.org/wiki/Trigonometric_functions

Figures 1 and 2: Circle perimeter on hyperbolic surfaces

The working of this section follows the method of UoG (n.dc). We consider a circle on a hyperbolic surface (Figure 3) where the shortest distance between any two points is a hyperbola. A triangle is shown as the first of an infinite number of equal area triangles to form an inscribed hyperbolic polygon to cover the circle.

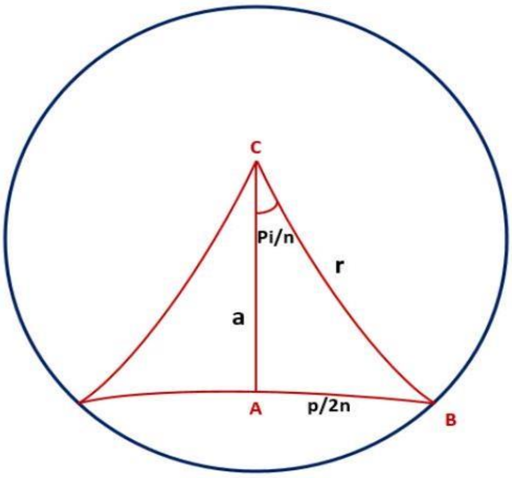

Figure 3 Showing a triangle on a hyperbolic surface where the shortest distance between any two points is a hyperbola.

Let p be the hyperbolic perimeter of an inscribed polygon consisting of n equal-area hyperbolic triangles, one of which is shown in Figure 3 . Let d be the area of the hyperbolic polygon (Weisstein, n.d).

We consider the polygon to be composed of 2n triangles CAB, each of area $\text{d}/2\text{n}$. We now use the well-known triangle sine rule for Δ CAB (Figure 3):

$$a/\sin A = b/\sin B = c/\sin C$$

which becomes for ΔCAB where $\sin \mathrm{A} = 1$, since A is a right angle:

$$\mathrm{a} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Thus we can writes

$$\sin C = \frac{c}{a}$$

which can be written for Figure 3 as:

$$\sin\left(\frac{\pi}{n}\right) = \sinh\left(\frac{p}{2n}\right)/\sinh(r) \qquad (1)$$

since x maps to $\sinh(x)$ on a hyperbolic surface where $\sinh(x)$ is the shortest distance between two points on the surface.

For n triangles, we can now write:

$$\mathrm{n}\sin\left(\frac{\pi}{n}\right) = \mathrm{n}\sinh\left(\frac{p}{2n}\right)/\sinh(r) \qquad (2)$$

To proceed further, we need the Limit Lemma.

Limit lemma

The working of this section also follows the method of UoG (n.db). Here we prove the limit lemma for the sin and sinh functions which states that:
If the function f is differentiable at 0 and $k \neq 0$, then: $nf(k/n) \to kf(0)$ as $n \to \infty$.

Proof: We use the well-known definition for differentiation.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

which, at $x = 0$, becomes:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

We can now replace h with x a position on the x-axis, since sin and sinh functions are both zero at $x = 0$, to derive:

$$f'(0) = \lim_{x\to 0} \frac{f(x)}{x}.$$

We now introduce constant $\mathrm{k} \neq 0$ such that $kx \to 0$ as $x \to 0$ Thus

$$f'(0) = \lim_{x\to 0} \frac{f(kx)}{kx}$$

and so

$$kf'(0) = \lim_{x\to 0} \frac{f(kx)}{x}.$$

Now let $k = 1/\mathrm{n}$, with $n \to \infty$ as $x \to 0$ then

$$nf'\left(\frac{k}{n}\right) \to kf'(0) \text{ as } n \to \infty.$$

For $f(x) = \sin x$ or $f(x) = \sinh x$ we have:

$$f'(0) = \cos 0 = 1$$
$$f'(0) = \cosh 0 = 1$$

so we can conclude:

$$f\left(\frac{k}{n}\right) \to k \text{ as } n \to \infty. \quad (3)$$

Incorporating the limit lemma

We can now proceed using (3). As $n \to \infty, p \to C(r)$, the hyperbolic perimeter, and so equation (2) becomes:

$$n\pi/n\sinh(r) \quad = n\frac{p}{2n}$$
$$\pi/\sinh(r) \quad = \frac{p}{2}$$
$$\text{or } p = 2\pi\sinh(r), \text{ and}$$
$$\pi\sinh(r) = \frac{C(r)}{2} \text{ with}$$
$$C(r) = 2\pi\sinh(r), \text{ the hyperbolic perimeter.}$$

Variation of Pi on a hyperbolic surface

We now consider the value of Pi for a circle on a hyperbolic surface and let π ' be the value of π (ratio of circumference to diameter) on a hyperbolic surface. We have:

$$\pi\sinh(r) = \pi' r$$

and so

$$\pi' = \pi\sinh(r)/r. \quad (4)$$

Variation of Pi on a spherical surface

The working of this section follows the method Maximenko (2015). We now consider a spherical surface and the Δ CAB (Figure 3).

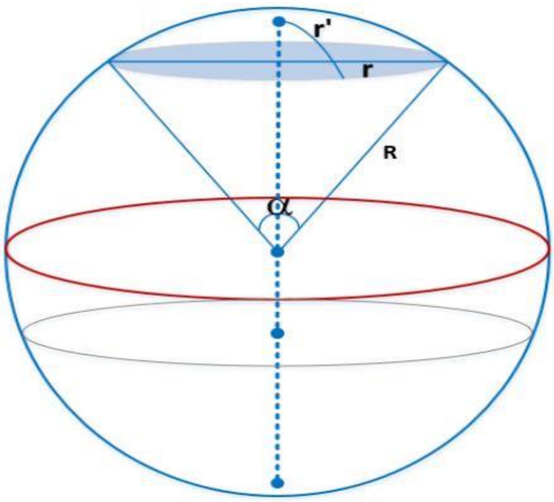

Figure 3 3D sphere with circles of surface radius r ' and plane radius r on its surface.

$$\text{Sin}\,\alpha/2 = r/R \quad (5)$$

from trigonometry and by definition of radian angle measure, we have:

Angle in radians = subtending arc / radius

$= \mathrm{r}'/\mathrm{R}$ for $\Delta\mathrm{CAB}$

$$= \frac{\alpha}{2} . \tag{6}$$

From (5) and (6) we can write:

$$\frac{r'}{R} = \frac{\alpha}{2} / \sin\left(\frac{\alpha}{2}\right)$$

and, since $r < r'$:

$$r/r' = \sin\left(\frac{\alpha}{2}\right) / \left(\frac{\alpha}{2}\right) < 1.$$

For a spherical surface, circle circumference is $2\pi r = 2\pi' r'$.

Where, again, π ' and r' refer to the surface values. Thus, we have an expression for the variation of π with surface radius:

$$\pi' = \pi r/r' < \pi$$

$$= \pi \sin\left(\frac{\alpha}{2}\right) / \left(\frac{\alpha}{2}\right)$$

$$= \pi \sin\left(\frac{r}{R}\right) / \left(\frac{r}{R}\right)$$

$$= \pi \sin(r)/(r) \tag{7}$$

for a sphere with unit radius, $R = 1$.

Comparing Pi variations

Using equations (4) and (7) we can compare the variation of Pi on each of the two types of surfaces discussed in this paper as the surface radius of circles is increased. Whilst the value of Pi on a plane surface is constant and does not vary with circle size, Pi increases rapidly for a hyperbolic surface and decreases for a spherical surface.

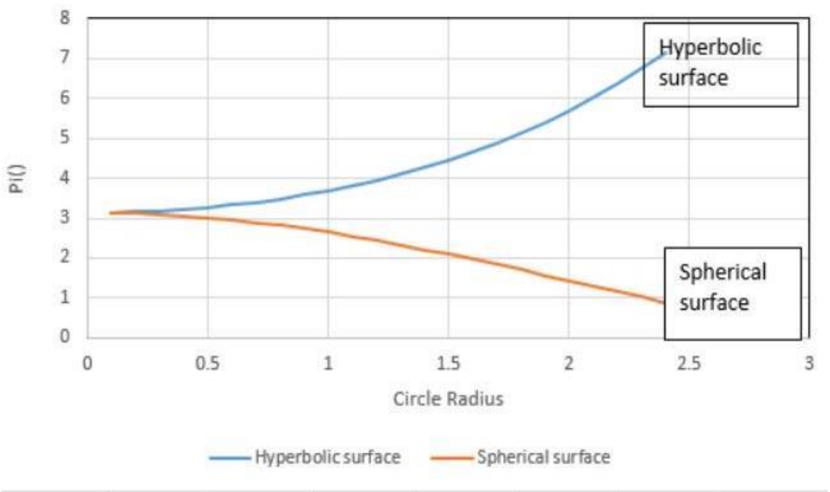

Figure 4 Using equations (4) and (7) we can compare the variation of Pi on each of the two types of surfaces discussed in this paper as surface radius is increased.

Surface area of a circle on a hyperbolic surface

Again, we consider a triangle on a hyperbolic surface (Figure 3) multiple occurrences of which will cover the whole circle area (UoG, n.da). From Heron's formula for triangle areas adapted for hyperbolic triangles:

$$\sin\left(\frac{d}{n}\right) = \sinh(a)\sinh\left(\frac{p}{2n}\right)/(\cosh((r) + 1)$$

where d is triangle area, and for 2n triangles we have:

$$2n\sin\left(\frac{d}{n}\right) = \sinh(a)2n\sinh\left(\frac{p}{2n}\right)/(\cosh((r) + 1).$$

Let D(r) and C(r) be the hyperbolic circle area and circumference respectively. As $n \to \infty$, $D(r) = d$, the hyperbolic circle area, $a \to r, \&$ and $C(r) = p$ the hyperbolic circumference. We can now write:

$$D(r) = (\sinh(r)C(r))/(\cosh(r) + 1)$$
$$= (\sinh(r)2\pi\sinh(r))/(\cosh(r) + 1)$$
$$= 2\pi\sinh^2\left(\frac{r}{2}\right)\cosh^2\left(\frac{r}{2}\right)/\left(2\cosh^2\left(\frac{r}{2}\right)\right)$$
$$= 4\pi\sinh^2\left(\frac{r}{2}\right)$$

Thus, area of a circle on a hyperbolic surface is given by:

$$D(r) = 4\pi\sinh^2\left(\frac{r}{2}\right). \tag{8}$$

Surface area of a circle on a spherical surface

The working of this section follows the method of StackExchange (2016). Referring to Figure 5 we have:

$$\alpha = \frac{r}{R} \text{ and } \sin\alpha = \frac{x}{R}$$
$$\sin\left(\frac{r}{R}\right) = \frac{x}{R}$$
$$x = R\sin\left(\frac{r}{R}\right).$$

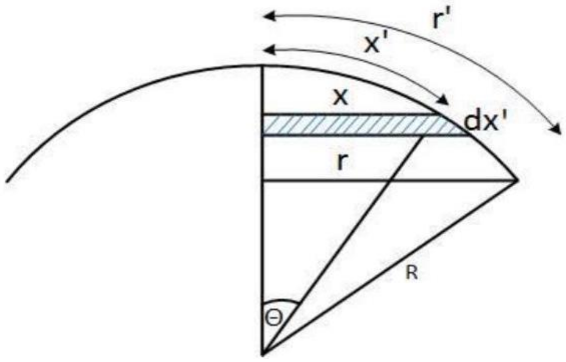

Figure 5 The area of a cap on a sphere is calculated by considering a thin disk of radius x and thickness dx' and then integrating the disk surface strip from 0 to r '.

Area of circular surface strip $= 2\pi x dx'$
Area of cap,

$$A = 2\pi R \int_0^{r'} \sin\frac{x'}{R} dx'$$

$$= 2\pi R^2 \int_0^{r'} \sin y dy \text{ where } y = x'/R \& R dy = dx' \quad = 2\pi R^2 [-\cos y]_0^{r'/R}$$

$$= 2\pi R^2 \left(1 - \cos\left(\frac{r'}{R}\right)\right)$$

$$= 2\pi(1 - \cos r')$$

for $R = 1$ on a unit sphere.
Thus, area of a circle on a spherical surface is given by:

$$D(r) = 2\pi(1 - \cos r'). \tag{9}$$

Comparing circle area variations

Using equations (8) and (9) we can compare the variation of circle area on each of the two types of surfaces discussed in this paper as the surface radius of circles is increased. Disks areas grow faster with length of surface radius on plane surfaces than on spherical surfaces and faster again on hyperbolic surfaces.

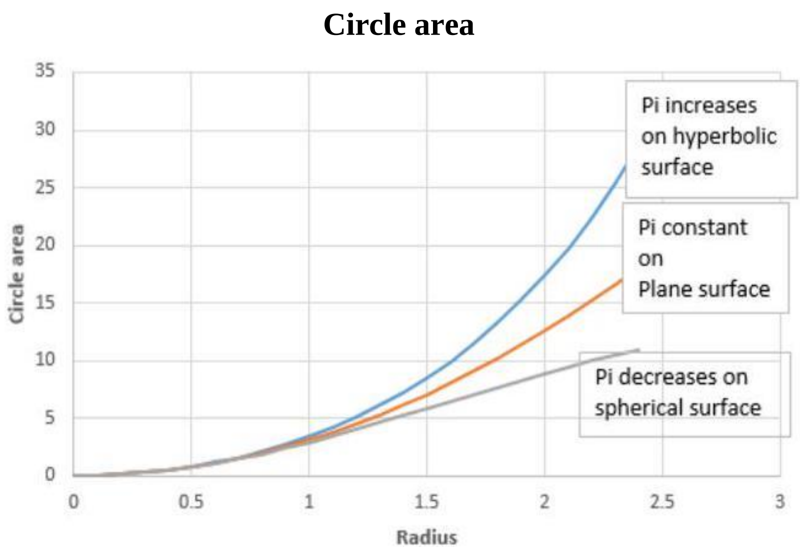

Figure 5 Disks areas grow faster with length of surface radius on plane surfaces than on spherical surfaces and faster again on hyperbolic surfaces.

Conclusion

This paper has assembled algebraic equations to demonstrate that the values of Pi decrease with increasing radius for surface circles on a sphere and increase on hyperbolic surfaces. This has been approached both by considering the ratio of circle circumference to the radius on each type of surface and also by considering the ratio of circle area to the radius on each surface.

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