Computing cube root of a real number

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Abstract

Computing square root of a real number of any digits (whether involving integral part only or both integral as well as decimal parts) is taught in lower classes in the Indian sub-continent but nowhere else outside. It has been a matter of great curiosity why computation of roots of higher order of a real number has never been taught? We formulated a method to compute the real cube root of a real number irrespective of its nature whether containing only an integral part or only decimal part or both.

Keywords: cube root, mathematical computation, real numbers.

§1. Cube root of four digits (whole) numbers which are perfect cubes of some integers

Let x_1, x_2, x_3 be the digits at the respective places of unity, ten, hundred assuming values from 0 to 9 and x_4 the digit at thousand's place (running from 1 to 9). Thus, writing the number as $x_4\, x_3\, x_2\, x_1$ its numerical value is

$$10^3.\, x_4 + 10^2.\, x_3 + 10.\, x_2 + x_1.$$

Rule 1.1. We group the digits of number starting from the unit's digit in triples, i.e. x_1, x_2, x_3 in the first group and the next group consists of the single digit x_4. Now, we look for the largest (whole) number *a* (say), whose *cube* does not exceed the digit x_4. As x_4 assumes values from 1 to 9, *a* can have only two choices: 1 and 2.

Next, we carry out the division of the given number ($x_4\, x_3\, x_2\, x_1$) by a^2 as illustrated below:

$$a^2 \left| \begin{array}{l} \overline{\quad x_4 x_3 x_2 x_1 \quad}^{\;a} \\ -a^3 \\ \hline x^4 - a^3 \equiv x_5 \\ \hline \end{array} \right. \qquad (1.1)$$

(say) to its right carry over the triple $x_3\, x_2\, x_1$ to get $x_5\, x_3\, x_2\, x_1$. Now, we choose the largest digit *b* (to be placed to the right of *a* in quotient) so that the cube of the number *a b* (with numerical value 10 *a* + *b*):

$$(10\, a + b)^3 \equiv 1000\, a^3 + (3 \times 100\, a^2 \times b + 3 \times 10\, a \times b^2 + b^3) \qquad (1.2)$$

does not exceed the given number. In other words, the number within the parentheses on RHS of Eq. (1.2), i.e. $\{3.(100\, a^2 + 10\, a.\, b) + b^2\}.b$ does not exceed the remainder $x_5 x_3 x_2 x_1$ obtained in Eq. (1.1). Thus, the second divisor

$$c \equiv 3.(100\, a^2 + 10\, a.\, b) + b^2 \qquad (1.3)$$

divides $x_5 x_3 x_2 x_1$ by b times:

$$3 \cdot (100\, a^2 + 10\, a \cdot b) + b^2 \left| \begin{array}{l} \overline{\quad x_5 x_3 x_2 x_1 \quad}^{\;ab} \\ -\{3(100\, a^2 + 10\, a \cdot b) + b^2\} \cdot b. \\ \hline \end{array} \right. \qquad (1.4)$$

If the remainder in the above division process in Eq. (1.4) is zero, the desired cube root is the number *a b* (with numerical value 10 *a* + *b*). //

Example 1.1. To compute the cube root of the number 1,728.

Soln. Proceeding as above, we carry the division. After the first process of division, we sought the digit *b* (to be placed to the right of quotient 1). Taking it as 1, the divisor *c* will be 3. $(100 \times 1^2 + 10 \times 1 \times 1) + 1^2 = 331$ whose product with *b* (i.e. 1) is much lower than 728. Repeating the exercise with $b = 2$, we get $c = 3.\ (100 \times 1^2 + 10 \times 1 \times 2) + 2^2 = 364$ whose product with $b = 2$ makes exactly 728.

$$
\begin{array}{r|r}
 & 12 \\ \hline
1^2 & 1,728 \\
 & -1 \\ \hline
364 & 728 \\
 & -728 \\ \hline
 & 0.
\end{array}
$$

Thus, the cube root of the given number is 12. //

Example 1.2. To compute the cube root of the number 15,625.

Soln. Proceeding as above, we carry the division. Beginning with $b = 4$, we get $c \equiv 3.\ (100{\times}2^2 + 10{\times}2{\times}4) + 4^2 = 1{,}456$ whose product with $b = 4$ is lower than 7,625. Repeating the exercise with $b = 5$, we get $c = 3.\ (100{\times}2^2 + 10{\times}2{\times}5) + 5^2 = 1{,}525$ whose product with $b = 5$ makes exactly 7,625.

$$
\begin{array}{r|r}
 & 25 \\ \hline
2^2 & 15,625 \\
 & -8 \\ \hline
1525 & 7625 \\
 & -7625 \\ \hline
 & 0.
\end{array}
$$

Thus, the cube root of the given number is 25. //

Example 1.3. To compute the cube root of the number 42,875.

Soln. Proceeding as above, we carry the division: 3 is the largest digit whose cube does not exceed 42. Thus, 3^2 divides 42 by 3 times leaving the remainder 15. Borrowing the triple 875 to the right of 15 we look for the largest digit *b* to be placed to the right of $a = 3$ in the quotient so that the next divisor *c*, given by Eqn. (1.3), divides the number 15,875 by *b* times. The choice of $b = 5$ giving $c = 3.\ (100 \times 3^2 + 10 \times 3 \times 5) + 5^2 = 3{,}175$ divides the number exactly by 5 times.

$$
\begin{array}{r|r}
 & 35 \\ \hline
3^2 & 42875 \\
 & -27 \\ \hline
3175 & 15875 \\
 & -15875 \\ \hline
 & 0.
\end{array}
$$

Thus, the cube root of the given number is 35. //

Example 1.4. To compute the cube root of the number 91,125.

Soln. Proceeding as above, we carry the division. Here, $a = 4$ is the largest digit whose cube does not exceed 91. Thus, 4^2 dividing 91 by 4 times leaves the remainder 27. Borrowing the triple 125 the next dividend is 27,125. Again, we look for the largest digit b to be placed to the right of 4 in the quotient so that the next divisor c, given by Eqn. (1.3), divides the number 27,125 by b times. The choice of $b = 5$ makes $c = 3.\ (100 \times 4^2 + 10 \times 4 \times 5) + 5^2 = 5{,}425$ that divides the number exactly by 5 times.

```
            45
      ----------
4^2   |  91125
      | -64
      | -------
5425  |  27125
      | -27125
      | -------
      |      0.
```

Thus, the cube root of the given number is 45. //

Example 1.5. To compute the cube root of the number 166,375.

Soln. Proceeding as above, we carry the division. Here, $a = 5$ is the largest digit whose cube does not exceed 166. Thus, 5^2 dividing 166 by 5 times leaves the remainder 41. Thereafter, borrowing the triple 375 the next dividend is 41,375. Exploring with $b = 4$ we would get $c = 3.\ (100 \times 5^2 + 10 \times 5 \times 4) + 4^2 = 8{,}116$ whose product with $b = 4$ is much lower than 41,375. But, $b = 5$ gives $c = 3.\ (100 \times 5^2 + 10 \times 5 \times 5) + 5^2 = 8{,}275$ whose product with $b = 5$ makes exactly 41,375.

```
            55
      ----------
5^2   | 166,375
      | -125
      | -------
8275  |  41375
      | -41375
      | -------
      |      0.
```

Thus, the cube root of the given number is 55. //

§ 2. Cube root of numbers having only decimal part but perfect cube

Let x_1, x_2, x_3, x_4 be the successive digits after the decimal towards the right forming the number $0{\cdot}x_1\, x_2\, x_3\, x_4$. Each one of the digits x_1, x_2, x_3, x_4 may assume values from 0 to 9. In this case, we group the successive digits x_1, x_2, x_3 in the first triple towards the right (unlike to Rule 1.1) and think of the largest (single digit) number, say a, with cube not exceeding the value of the number

$$10^2.\ x_1 + 10.\ x_2 + x_3 \equiv y_1 \quad \text{(say)}, \qquad (2.1)$$

formed by the digits x_1, x_2, x_3 and carry out the *first* process of division of the number y_1 by a^2 leaving quotient a and remainder $y_1 - a^3 = y_2$ (say). The second triple is formed by digit x_4 and additional two (borrowed) digits each of value 0 (zero), which is always permissible. This second triple with digits x_4, 0, 0 is placed to the right of remainder y_2 and the process of subsequent division(s), as explained above, after Eqn. (1.1), is followed. The noticeable point here is that the role of decimal is almost ignored. However, it is placed before *a* in the quotient.

The process is elaborated through the following examples.

Example 2.1. To compute the cube root of the number 0·729.

Soln. The largest single digit with a cube not exceeding the number 729 (ignoring the decimal) is 9. Thus, taking $a = 9$, the division is carried by $a^2 = 81$:

$$\begin{array}{r|r} & 0\cdot 9 \\ \hline (0.9)^2 = 0\cdot 81 & 0\cdot 729 \\ & -0\cdot 729 \\ \cline{2-2} & 0 \\ \cline{2-2} \end{array}$$

determining the desired cube root 0·9. //

Example 2.2. To compute the cube root of the number 0·001331.

Soln. The first triple to the right of the decimal point is 001; so, here, $a = 1$. Division by $a^2 = 1$ leaves the quotient 1 and remainder 0. The next triple 331 is placed to the right of remainder 0 (forming the number 331). Now, we think of the single (largest) digit *b*, to be placed to the right of existing digit $a = 1$ in the quotient and the next divisor [cf. § 1]

$$c \equiv 3.\,(100\,a^2 + 10\,a.b) + b^2 \;=\; 3.\,(100 \times 1^2 + 10 \times 1 \times b) + b^2$$

divides 331 by *b* times. Such a choice of *b* is only 1 and the corresponding value of $c = 331$ divides the number 331 by once. Hence, the digits $a = 1$ and $b = 1$ determine the cube root 0·11.

$$\begin{array}{r|r} & 0\cdot 11 \\ \hline (0.1)^2 = 0\cdot 01 & 0\cdot 001\,331 \\ & -0\cdot 001 \\ \cline{2-2} 331 & 331 \\ & -331 \\ \cline{2-2} & 0. \\ \cline{2-2} \end{array}$$

Example 2.3. To compute the cube root of the number 0·015625.

Soln. The first triple to the right of the decimal point is 015; so, here, $a = 2$. Division by $a^2 = 4$ leaves the quotient 2 and remainder 7. The next triple 625 is placed to the right of remainder 7 (forming the number 7,625). Now, we think of the single (largest) digit *b*, to be placed to the right of existing digit $a = 2$ in the quotient, so that the next divisor *c* in Eqn. (1.3) may divide the number 7,625 by *b* times. Such a choice of *b* is 5, and corresponding $c = 1{,}525$ exactly divides 7,625 by 5 times. Hence, the desired cube root is 0·25.

$$
\begin{array}{r|r}
 & 0\cdot 25 \\
(0.2)^2 = 0\cdot 04 & 0\cdot 015625 \\
 & -0\cdot 008 \\
1525 & 7625 \\
 & -7625 \\
 & 0.
\end{array}
$$

§ 3. Cube root of any whole number

The process is the same as explained in §1 and we continue up to Eqn. (1.4). Naturally, the remainder in Eqn. (1.4) is not *zero* here. Hence, we place a decimal point to the right of (given whole number) and borrow a triple formed by digits each of value 0 (zero) to be placed after the remainder. Simultaneously, a decimal point is also placed immediately in the quotient. Thereafter, the process as explained in §2 is followed. We demonstrate the process using some examples.

Example 3.1. To compute the cube root of the number 11.

Soln. Since there are only 2 digits in the integral part of the number we do not try to form a triple and just think of a single digit largest number with a cube not exceeding 11. Such a digit is 2. Thus, taking a = 2, the first process of division is carried by a^2 = 4 leaving the quotient 2 and the remainder 3. Thereafter, decimal points are simultaneously placed in the quotient as well as to the right of (given number) 11. Now, we borrow triple (000) with each zero digit and place it to the right of remainder 3 to form the dividend 3,000. We think of the largest single digit b to be placed in the quotient (after the decimal point) so that the corresponding value of next divisor c, given by Eqn. (1.3), may divide the number 3,000 by b times. Such a choice of b is 2 so that c = 1,324 divides 3,000 by 2 times leaving the remainder 352. Again borrowing the next triple 000 and placing it to the right of remainder 352 making the dividend 352,000. We think of the next single largest digit b to be placed to the right of 2·2 in the quotient, so that the corresponding c may divide 352,000 by b times. Noting the current value of a as 22 (ignoring the decimal point in the quotient), b = 2 again meets the requirements and

$$c \equiv 3.\,(100 \times 22^2 + 10 \times 22 \times 2) + 2^2 = 146{,}524$$

divides 352,000 by 2 times, leaving the remainder 58,952. Again, borrowing the next triple 000 after the above remainder makes the dividend 58,952,000. Now, with a = 222, we look for the largest b so that the corresponding c can divide this new dividend. The choice b = 3 meets the requirements and

$$c \equiv 3.\,(100 \times 222^2 + 10 \times 222 \times 3) + 3^2 = 14{,}805{,}189$$

divides 58,952,000 by 3 times, leaving the remainder 14,536,433.

	2 · 223
2^2	11.000 000 000
	− 8
1324	3000
	−2648
146524	352000
	−293048
14805189	58952000
	−44415567
	14536433.

Thus, computing up to 3 decimal places, the desired cube root is 2·223. //

§ 4. Cube root of a mixed number having both integral and decimal parts

For the cube root of an integral part, the process is the same as explained in §1 and we continue up to Eqn. (1.4). Again, the remainder in Eqn. (1.4) may not be *zero* in general. Hence, we place a decimal point to the right of the available quotient and borrow the available triple formed by the first three digits after the decimal point in the dividend to the right of the remainder. Thereafter, the process as explained in § 2 is followed.

We demonstrate the process using some examples.

Example 4.1. To compute the cube root of the number 4·096.

Soln. Since the integral part of the number has only one digit we ignore the concept of a triple and just think of a single digit largest number with a cube not exceeding the integral part. Such a digit is 1. Thus, taking a = 1, the first process of division is carried by a^2 = 1 leaving the quotient 1 and the remainder 3. Thereafter, a decimal point is placed in the quotient and the triple (096) available in the decimal part of the number is carried to the right of remainder 3 to form the dividend 3,096. We think of the largest single digit b to be placed in the quotient (after the decimal point) so that the corresponding value of next divisor c, given by Eqn. (1.3), may divide the number 3,096 by b times. Such a choice of b is 6 so that c = $3.(100\times1^2 + 10\times1\times6) + 6^2 = 516$ divides 3,096 by 6 times exactly leaving zero remainder.

	1 · 6
$1^2 = 1$	4 · 096
	− 1
516	3096
	−3096
	0.

Hence, the desired cube root is 1·6. //

Example 4.2. To compute the cube root of the number 1234·5678.

Soln. Following the method explained in §1, the first triple (counted from the decimal point towards left) is 234 and thereafter, there is only a single digit 1. As such, there exists only a number 1 whose square divides this single digit dividend 1 by 1. So, the first available digit in the quotient is also 1 and the remainder is zero. Thereafter, the next triple (234) is borrowed to the right of remainder 0 to form the next dividend 0234, i.e. 234 only. Looking for the largest single digit *b* to be placed in the quotient so that the corresponding *c* given by Eqn. (1.3) divides 234 by *b* times. Such choice of *b* is only 0 for which $c = 300$ divides 234 by 0 times leaving the remainder again 234 and the quotient becomes 10. Thereafter, a decimal point is placed in the quotient and the triple (567) available in the decimal part of the number is borrowed to the right of remainder 234 to form the new dividend 234,567. The search for the next largest single digit *b* to be placed in the quotient (after the decimal point) is made so that the corresponding value of divisor *c* may divide 234,567 by *b* times. Such a choice of *b* is 7 making the quotient 10·7 and $c = 32{,}149$ that divides 234,567 by 7 times leaving the remainder 9,524. Again borrowing the next triple 800 and carrying it to the right of the remainder 9,524 making the dividend 9,524,800. Again, the next search for the single largest digit *b* to be placed to the right of the quotient 10·7, so that the corresponding *c* may divide 9,524,800 by *b* times. Noting the current value of *a* as 107 (ignoring the decimal point in the quotient), $b = 2$ meets the requirements and $c \equiv 3.\,(100 \times 107^2 + 10 \times 107 \times 2) + 2^2 = 3{,}441{,}124$ divides 9,524,800 by 2 times, leaving the remainder 2,642,552. The quotient gets supplemented as 10·72. Borrowing the next triple 000 to the right of the above remainder makes the dividend 2,642,552,000. Now, with $a = 1{,}072$, we look for the largest *b* so that the corresponding *c* can divide this new dividend. The choice $b = 7$ meets the requirements supplementing the quotient as 10·727 and $c \equiv 3.\,(100 \times 1072^2 + 10 \times 1072 \times 7) + 7^2 = 344{,}980{,}369$ divides 2,642,552,000 by 7 times, leaving the remainder 227,689,417.

	10 · 727
1^2	1234 · 567 800 000
	− 1
300	234
	−0
32149	234567
	−225043
3441124	9524800
	− 6882248
344980369	2642552000
	− 2414862583
	227689417 .

Thus, computing up to 3 decimal places, the desired cube root is 10·727. //

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