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Electronic Journal of Informatics

The Electronic Journal of Informatics provides a platform for the staff and students (current and past) of the Faculty of Business and Informatics at the Divine Word University (DWU) to publish their work. The Journal is open for articles in the areas including business and management, information and communications technology, mathematics and computing science, and tourism and hospitality.

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Editors

Peter Anderson (PhD) is a Professor and Head of Departments of Information Systems and Mathematics & Computing Science at DWU. He is particularly interested in mathematical modelling and mentoring young academics in both teaching and research. Email: panderson@dwu.ac.pg

Martin Daniel (PhD) is an Associate Professor and the Dean of the Faculty of Business & Informatics at DWU. He holds a Bachelor's degree in Information Technology from the University of Queensland in Australia, a Master's degree in Information Technology from the Queensland University of Technology in Australia and a Doctor of Philosophy from the Divine Word University. His interests include researching and modelling technological developments (e.g. e-government and e-learning) in PNG. Email: mdaniel@dwu.ac.pg and mdmartindaniel@gmail.com

Vanessa Uiari is a senior lecturer in the Department of Tourism and Hospitality Management. She has been employed with DWU since 8 February 2008. Her research interest is in critical tourism studies in the South Pacific and Oceania. She conducted an ethnographic study of the Isurava people of Iora Creek valley along the northern stretches of the Kokoda Trail in Oro province

for her doctoral study, which she hopes to complete in 2020. Email: vuiari@dwu.ac.pg

Cyril Sarsoruo is a lecturer in the Department of Mathematics and Computing Science at DWU. He holds a Master's degree in Theoretical Mathematics and specializes in mathematical analysis. His research interests include functional equations and inequalities. Email: csarsoruo@dwu.ac.pg

Raunu Gebo Sarsoruo is a lecturer in the Department of Mathematics & Computing Science at DWU. She holds a Master's Degree in Mathematical Methods for Computing Science from the University of Silesia in Poland and specialises in pure and applied mathematics. Her research interest includes applying different mathematical concepts through programming all kinds of computer systems and developing mobile applications to solve real-world problems. Email: rgebo@dwu.ac.pg

Information for contributors

Articles should be 3000 – 6000 words in length including the abstract and references. Research note, book review and editorial should be 900 – 1500 words. A no-more-than 100 words biographical note about the author should accompany the article. Full information for authors can be accessed at https://www.dwu.ac.pg/en/images/All_Attachements/FBI/e-Journals/E-Journal-guide-for-authors.pdf

Editorial note

Welcome to Volume 3 of the Electronic Journal of Informatics, which is an annual publication produced by the Faculty of Business and Informatics, Divine Word University. Publication of this journal began in 2019 to promote research culture through intensified research and knowledge exchange from its contributors.

Martin Daniel discusses the manual process of selecting grade twelve school leavers in Papua New Guinea and its challenges, the online selection process and its benefits, and provides some suggestions for further improvement and the optimum use of the online system.

Allan Sumb presents the benefits of mega-events such as Asia Pacific Economic Cooperation, APEC, summit in a host country, contributing to economic development. These benefits include (1) APEC was seen to provide an opportunity for PNG to showcase a positive destination image to the world, (2) the infrastructures built for APEC will remain as legacies of mega-events and continue to benefit PNG and (3) APEC helped the growth of the tourism and hospitality industry.

Peter Anderson presents some issues in solving non-linear polynomial equations. He shows that a simple geometrical problem generates a degree 8 polynomial function after firstly applying Pythagoras' Theorem and then squaring the resulting equation to derive a more elegant polynomial equation. The degree 8 polynomial has 4 real and 4 complex solutions as expected. Squaring introduces extraneous solutions and only one of the final 8 solutions solves the Crossed Ladders problem.

Raunu Gebo Sarsoruo presents differentiability in normed spaces and shows a new approach using common abstractive ideas to develop a better understanding of differentiation. She shows that foundational concepts from limits that relate to continuity, then to linearity and bilinearity in the form of definitions, theorems and lemmas including some of their proofs provide a better way of understanding differentiability in calculus.

Cyril Sarsoruo presents random variables and convex functions in Stochastic orderings with its applications in mathematics. He shows the basic relationship of probability to random variables followed with convex function and formulations of theorems using the Ohlin lemma in their proofs.

Rik King and **Peter Anderson** present Monte Carlo simulations to estimate Pi. They show that a simple Monte Carlo simulation, using functions from various R packages can be used to calculate π using a variety of polygons to circumscribe a unit circle.

Hope that you find the articles interesting and informative.

Associate Professor Dr Martin Daniel (PhD)
Coordinator and Chief Editor of the Journal

National online selection system in Papua New Guinea

Martin Daniel

Abstract

Many countries around the world are using information and communication technologies to improve their educational processes, thereby, attaining a significant reduction in time, cost and other resources, leading to effective and efficient educational service delivery. The Papua New Guinea (PNG) Department of Higher Education, Research, Science and Technology (DHERST) in partnership with PCG Academia developed an online system to improve the process of selecting suitable grade twelve school leavers applying to study at higher educational institutions. This paper provides a brief background of PNG higher education and DHERST. It also discusses the manual process of selecting school leavers and its challenges. The paper then discusses the online selection process (national online selection system) and its benefits. Lastly, it provides some suggestions for improvement and optimum use of the online selection system.

Keywords: admission pool, backup choices, educational services, algorithm, grade twelve school leavers (SLs), grace period, higher education, Information and Communication Technology (ICT), manual selection process, National Online Selection System (NOSS), online selection process, Papua New Guinea (PNG) National Department of Education (NDoE), PCG Academia, PNG National Department of Higher Education, Research, Science and Technology (DHERST).

Introduction

Many countries are using information and communications technologies (ICT) to improve their educational processes, thereby gaining a significant reduction in time, cost and other resources, and leading to effectiveness and efficiency in delivering educational services. Many developing countries such as Papua New Guinea (PNG) are striving to adopt technological innovations to gain from the benefits of such innovations.

This paper will provide a brief background of higher education in PNG and DHERST. It will also discuss the manual process of selecting grade twelve school leavers (SLs) and its challenges. The paper will then discuss the online selection process (National Online Selection System, NOSS) and its benefits. Lastly, it will provide some suggestions for further improvement and the optimum use of the NOSS. The next section will now provide the background including the PNG Department of Higher Education, Research, Science and Technology (DHERST).

Higher education in Papua New Guinea

Higher education in PNG is an important sector for national socio-economic development (DHERST, 2018c). Providing quality educational services could contribute to achieving some of the stated aspirations of the PNG Government.

DHERST "is the government agency responsible for coordinating higher and technical education and research, science and technology in [PNG]. [It works with various] agencies and other partners to provide the advanced skills, knowledge and innovation required for PNG's sustainable social, economic and environmental development. [Its] vision [is] to serve and facilitate developments in the … higher and technical education sector for economic, social growth and nation-building. [Its] mission [is] to provide the best policy direction and support systems to enhance the quality, access and relevance of higher and technical education and research....” (DHERST, 2018d, para. 1 - 3).

DHERST’s strategic plan has been developed “to guide the transformation, reformation and unification of the structure of higher education…. [It aims] “to improve the quality of education on offer in HEIs [Higher Educational Institutions and] that this will directly benefit the social and economic development….” (DHERST, 2015, para. 5 - 6). DHERST also aims “to help as many students as possible to gain entry to universities or colleges, to offer the quality of education and...help them succeed once they have enrolled. It means that our hardest work needs to involve finding the strategies…that will best enable students to meet their educational goals” (DHERST, 2018a, para. 2).

DHERST’s activities include the selection of SLs to study at HEIs. It seeks to improve the process, thereby, ensuring quality and equality selection. It also aims “to improve transparency, accountability, and most importantly increase the probability of capable and eligible school leavers in being admitted while ensuring [that] institutions’ autonomy in [the] selection process is maintained” (Papua New Guinea Today, 2017, para. 5). Therefore, an initiative was undertaken in 2017 to develop an online selection system to ensure optimal matching of candidates and programs of study at HEIs (DHERST, 2018c). This system was developed to improve the manual selection process. The next section will now discuss the manual selection process and its challenges.

Manual selection process

Before 2017, the whole process of selecting SLs for a program of study at their preferred HEIs was performed manually. With the manual process, a team of selectors from various institutions travelled to Port Moresby (PNG’s capital city) to select suitable SLs for programs of study at their preferred institutions based on their choices (PCG Academia, 2017). “During the national selection, all HEIs select from candidates [SLs] who indicated their programs as [the] first choice, GPA being the main selection criteria. After the first round of selection, if there were some places left in the HEIs’ programs, [the] selectors added candidates [who] selected a particular institution as their second choice and later possibly from candidates who had indicated that they would be willing to accept other offers from similar fields of study” (DHERST, 2018c, para. 2) (Figure 1).

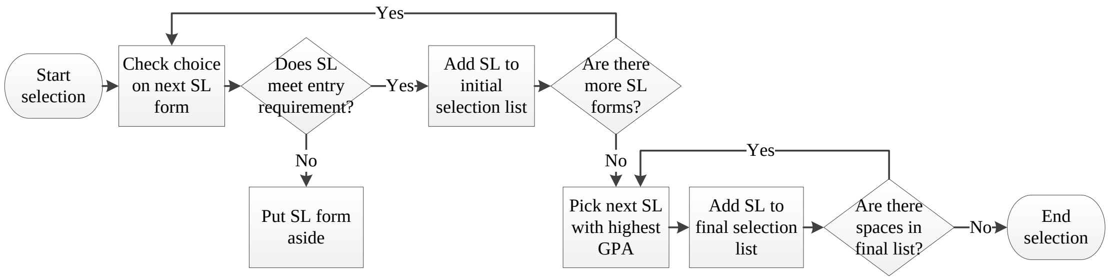

Figure 1: A simplified process model that shows how the selection of SLs is conducted manually based on the SL’s GPA and HEI program requirements.

The manual selection process had several issues (Figure 2). It was very slow, time-consuming, costly, and labour-intensive for DHERST and the HEIs. The manual process required a significant amount of resources and usually took several days to complete the selection. Further, the process led to suboptimal matching between SLs and HEIs’ programs. Many SLs often under or overestimated their academic potential, and “applied for programs for which they did not meet academic requirements…[and] lost their chances to be admitted to the less selective program. Some candidates, who passed exams with high GPA, apply for easier programs. Furthermore, manual selection of first and second choice candidates often led to missing better candidates” (DHERST, 2018c, para. 3). The process was slow in getting the results to the HEIs and selected SLs. Further, there was also a perceived lack of transparency and nepotism.

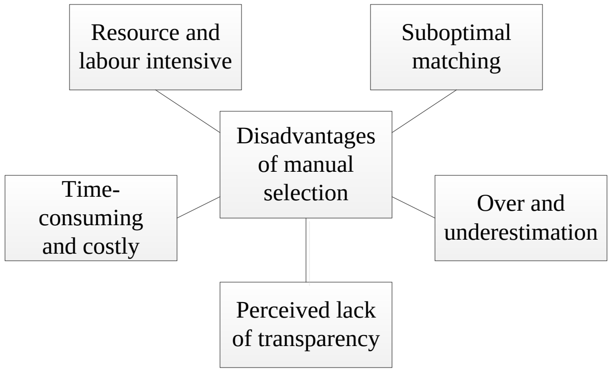

Figure 2: Some issues with the manual selection process

The manual process did not guarantee that all suitable SLs were considered with an equal chance of selection to a program of study at their preferred institutions. This led to many SLs not being selected in the manual process (DHERST, 2018c; PCG Academia, 2017). The above issues led to the development of the NOSS to improve the selection process, which will now be discussed.

National online selection system

In 2017 (as previously mentioned), DHERST undertook an initiative to develop the NOSS through a partnership with PCG Academia (DHERST, 2018c; Kora, 2017; Papua New Guinea Today, 2017; Post Courier, 2017a, 2017b). The NOSS was developed to improve the selection process and increase the chances of selecting suitable SLs applying to study at HEIs. In doing so, DHERST aimed to improve transparency, accountability, timeliness, efficiency and cost-effectiveness of the admission, verification and selection process (DHERST, 2018b; Kora, 2017; Post Courier, 2017b).

In December 2017, DHERST launched the NOSS, which was developed by the PCG Academia (PCG Academia, 2017). PCG Academia has a vast experience in developing information systems for student admission and management in various institutions around the world (DHERST, 2018c; Papua New Guinea Today, 2017; Post Courier, 2017a, 2017b). The first online selection resulted in about 47 per cent of all the SLs being selected to a program of study at an HEI (PCG Academia, 2017).

The NOSS considers all the SLs and selects suitable candidates by matching their choices (programs of study and their preferred HEIs) and grades against the program requirements of the preferred HEIs (PCG Academia, 2017; Post Courier, 2018) (Figure 3 & Figure 4). The system selects SLs to a study program at their preferred institutions based on their choices, grades and other requirements (e.g. quota) set by the HEIs (Loop PNG, 2018; Post Courier, 2018). These requirements are necessary to ensure that suitable SLs are selected (Study in PNG, 2019). The system is transparent, efficient and cost-effective. It also increases the chances of capable candidates being selected (Kora, 2017; Loop PNG, 2018) and ensures transparency, fairness and unbiasedness (Study in PNG, 2019).

The autonomy of the HEIs is maintained whereby the institutional requirements such as the program entry requirements, quota and other requirements are provided by the HEIs. The choices are provided by the SLs through the school leaver forms (SLF) submitted via the National Online Application System (NOAS) while their grades are provided by the PNG National Department of Education (NDoE) (PNG Insight, 2019; Study in PNG, 2019) (Figure 3). The NOSS uses this information to perform the selection and generate a list of suitable candidates for their program of study at their preferred institutions (Study in PNG, 2019) (Figure 4).

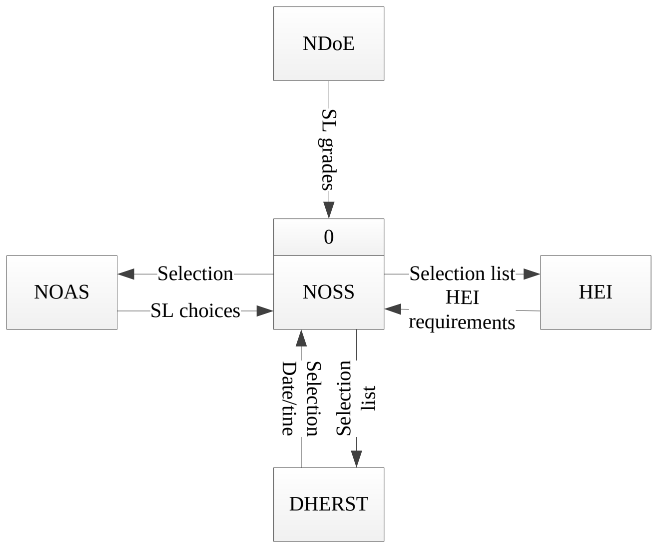

Figure 3: A simplified model (context diagram) showing the inputs (SL choices, SL grades and HEI requirements) and output (selection list) of an online selection system.

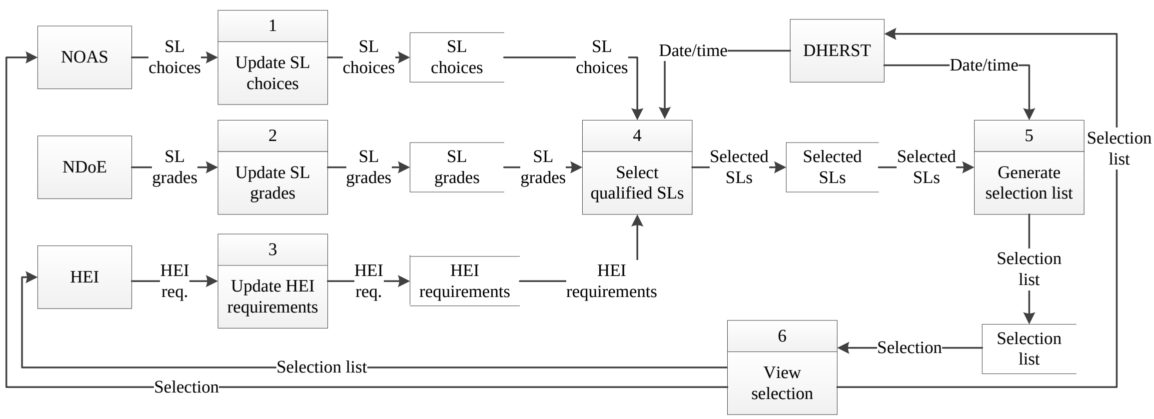

Figure 4: A model showing inputs (e.g. choices and grades), internal processes (e.g. Update choices and Perform selection) and outputs (e.g. selection list) of an online selection system. SLs (via NOAS), HEIs and NDoE provide inputs and view the selections generated by the system while SLs can view their selection using the NOAS.

The NOSS uses an algorithm, which compares the SLs' parameters (grades and choices) against the HEI requirements, performs the selection and generates a list of suitable candidates (DHERST, 2018c; PCG Academia, 2017). Apart from the grades, the HEIs may have other requirements, which are usually made known to DHERST, the schools and their students (SLs) (Study in PNG, 2019).

When the NOSS was developed, awareness about the NOSS was provided through various means such as media. Trainings and workshops were conducted to train the selectors (HEI staff) on how to use the system (PCG Academia, 2017; Study in PNG, 2019). Seminars were provided to the stakeholders including the HEIs to show how the system works (Papua New Guinea Today, 2017; Post Courier, 2017a, 2017b). A helpdesk was also established in DHERST to attend to queries regarding the online selection (Post Courier, 2018).

Some concerns were raised by some SLs, who claimed to have had good grades or met the program entry requirements but were not selected by the NOSS. For example, an SL claimed to have scored an A in a subject and Bs in other subjects and applied for a business-related program at an HEI but was not selected (Nazel, 2018). An SL may have not been selected due to several reasons including (1) the program quota may have been reached by the already selected SLs with higher or similar grades or (2) the choices may have not been prioritized properly, with the SLs overestimating their academic potential (Post Courier, 2018).

Several strategies were implemented to ensure fair and quality selection, giving the SLs with good grades equal chances of being selected (Figure 5). The NOSS provides other options, institutions that offer similar programs, to ensure that the SLs are treated fairly and justly without any biases, giving equal chances of being selected (Post Courier, 2018).

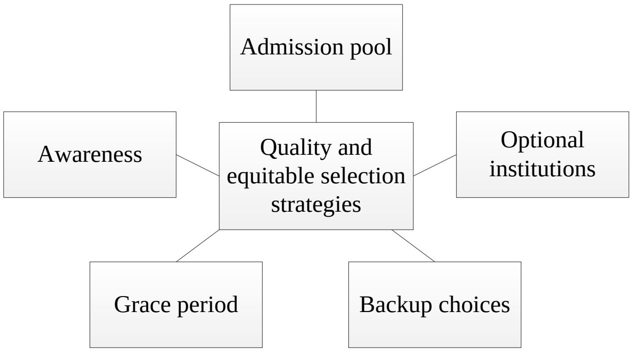

Figure 5: Several strategies were implemented to ensure quality and equitable selection of students with good grades.

An admission pool was implemented to contain the SLs who are not selected, placed according to their academic performance, but meet the entry requirements of a program of study. The institutional selectors can access the admission pool and manually fill in spaces in their programs usually after the national selection or during registration when the selected SLs fail to register at the beginning of the academic year (Study in PNG, 2019). The SLs who are not selected are also advised to approach the institutions for consideration from the admission pool (Study in PNG, 2019).

Moreover, the SLs are given a grace period of about 72 hours to make any changes to their choices through the NOAS before the automatic national online selection is conducted (Post Courier, 2018; Study in PNG, 2019). The SLs can access the NOAS using their usernames and passwords provided by DHERST. Usually, at this time, the SLs would have known their grades. Within this grace period, the SLs and their parents have adequate time to make any necessary well-informed choices. After the grace period, the automatic selection is performed and the SLs can see immediately whether they have been selected or not.

Further, backup choices were implemented in the NOAS to allow SLs to be selected into different study programs if they were not selected in their top five choices (Post Courier, 2018; Study in PNG, 2019). These backup choices could be similar to their preferred study programs or those for which they have met the requirements.

NOSS benefits

The NOSS has several benefits (Figure 6), which are discussed below.

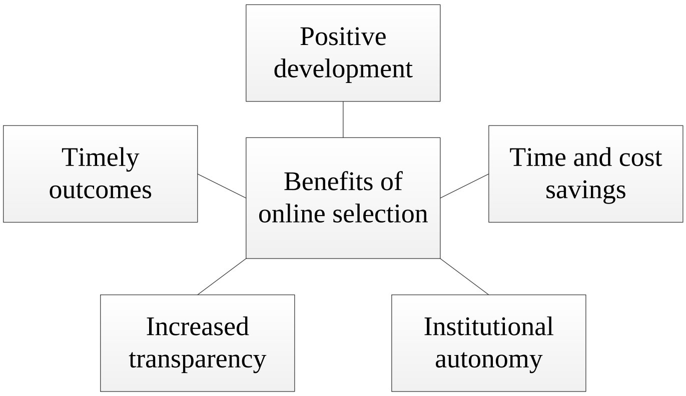

Figure 6: Benefits of the online selection process (NOSS)

Positive development

NOSS is seen as a timely initiative that is “practical and sustainable…to cater for growing school population in the country” (PNG Insight, 2019, para 47). It is a positive development, enabling a fair, efficient and cost-effective process (Owa, 2019; Sefe, 2019). The HEIs experienced significant differences between the manual process and the online process, which had a positive impact on academic operations and capable students being selected by the NOSS (Sefe, 2019).

Time and cost savings

NOSS enables significant cost and time savings unlike the manual process, which required costs of travel, accommodation and allowances for the institutional selectors and hiring of a venue for the selection (Owa, 2019; Post Courier, 2017a, 2017b; Sefe, 2019). With the NOSS, the selectors do “not have to spend time away from their normal duties” (Owa, 2019, para. 6).

Increasing transparency

NOSS uses a proven algorithm in the selection process, ensuring transparency and lucidity, preventing unfairness, favouritism and corruption perceived to have occurred in the manual process (Kora, 2017; Owa, 2019; Post Courier, 2017a, 2017b; Sefe, 2019). The online selection process is fair and unbiased (PCG Academia, 2017). Each child has the right to education and the NOSS gives the SLs equal consideration as long as the requirements are met (Post Courier, 2018).

NOSS was developed to improve transparency, accountability and efficiency in the selection process. The manual process was perceived to have had high instances of dubious selections, which resulted in SLs being selected through nepotism (e.g. wantok system) (The National, 2018). NOSS ensures transparency, prevents nepotism and unfair practices that would have been possible in the manual process (PCG Academia, 2017). The former Minister for DHERST, Mr Niningi, assured that “there will no longer be bribery, nepotism and corrupt practices. It will be a just, fair and quality education system that gives every individual child the right to education” (Post Courier, 2018, para. 11).

Timely outcomes

The universities receive their selection lists on time, giving them adequate time to prepare offer letters to be sent to the selected SLs on time unlike in the past where institutions waited long periods to receive their selection lists (Sefe, 2019). Further, the SLs will know immediately whether they have been accepted into an institution or not (PCG Academia, 2017). In the past, the SLs and their parents would wait, without knowing whether they (SLs) have been selected or not. If selected, they would wait for long periods to receive their offer letters, giving limited time to raise school fees or seek sponsorship. Parents and students can view their outcomes immediately after the online selection using the NOAS, giving them adequate time to prepare for higher education (Post Courier, 2018).

Institutional autonomy

The HEIs still maintain their autonomy in the selection process by setting their requirements. DHERST only facilitates the selection process and supports the HEIs to select suitable candidates without any undue influence (Sefe, 2019). The NOSS increases the chances of capable SLs being selected while still maintaining the autonomy of the institutions (The National, 2018).

The next section provides some suggestions for the optimum use of the NOSS.

Suggestions for optimum use

Regular open dialogue and proper communication between DHERST and all the HEIs need to be maintained (The National, 2018). Such communication is required to ensure that the HEI requirements are correct, accurate and up-to-date in the NOSS. These requirements also need to be communicated well and clearly to the schools, principals, guidance officers, SLs and their parents.

The SLs need to make well-informed decisions when making their choices using the NOAS. They must access their results on time to update their choices or make any necessary changes to increase their chances of being selected to a program of study at their preferred institution. The SLs need to make use of the strategies discussed earlier (e.g. backup choices and grace period).

The schools need to provide relevant information about the HEI program offerings and requirements to the SLs. They need to provide proper guidance to their students in the application process when using the NOAS. The SLs may also need to discuss with their parents, who might be able to provide some advice when making their choices.

Further, DHERST needs continuous support from the schools, SLs, principals and guidance officers, teachers and parents, HEIs, NDoE and other relevant stakeholders. They all need to continue working together to ensure that the NOSS is used properly to ensure a quality selection that is fair and transparent and gives equal chances of selection of suitable SLs.

Ongoing awareness about how the NOSS works, its features (e.g. backup choices, and admission pool) and benefits also need to be provided through various forms of media (e.g. social media, EM TV, TV WAN and NBC). Required training may also need to be provided to HEIs, especially for selectors, who may not know how to use the system. Such awareness and training will avoid misunderstandings and misconceptions about the NOSS.

Conclusion

Technological advancement has enabled many countries to improve their educational processes and deliver services with increased effectiveness and efficiency. The PNG Government through DHERST, in partnership with PCG Academia, developed the NOSS to improve the process of selecting SLs for further studies at HEIs.

This paper discussed the role of higher education and DHERST, manual selection process and its disadvantages, online selection and its benefits, and some suggestions for further improvement and optimum use of the NOSS.

Higher education is an important sector for national socio-economic development. THE PNG Government, through DHERST, aims to provide quality educational services and contribute to achieving the stated Government aspirations.

The previous manual process of selecting SLs had several issues. It was time-consuming, slow, costly and resource-intensive. The manual process was perceived to have a lack of transparency and involved nepotism, leading to high chances of questionable selection of SLs. These issues led to the development of the NOSS to implement the online selection process.

The NOSS was implemented to improve the selection process and achieve increased efficiency and effectiveness. It aimed to reduce cost, time and resource-intensity, eliminate nepotism and increase transparency, leading to a fair selection where all suitable SLs have an equal chance of being selected.

The NOSS is a timely and positive development and expected to contribute towards PNG's socio-economic development. It is, therefore, important to consider the provided suggestions for further improvement and optimum use of the NOSS.

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Future paper

There was limited literature about the national online selection system at the time of writing this paper. Hence, a future paper could discuss other aspects of the system, which are not discussed in the paper, as more literature becomes available.

Acknowledgement

I would like to thank Professor Peter K. Anderson for reviewing this paper. However, responsibility for any errors of fact or opinion remains with the author.

Author

Martin Daniel (PhD) is an Associate Professor and the Dean of the Faculty of Business & Informatics at DWU. He holds a Bachelor's degree in Information Technology from the University of Queensland in Australia, a Master's degree in Information Technology from the Queensland University of Technology in Australia and a Doctor of Philosophy from DWU. His interests include researching and modelling technological developments (e.g. e-government and e-learning) in PNG. Email: mdaniel@dwu.ac.pg and mdmartindaniel@gmail.com.

Benefits of mega-events: The Asia Pacific Economic Cooperation meeting in Papua New Guinea

Allan Sumb

Abstract

Mega-events create a powerful destination image for the host nation. This paper presents the benefits of mega-events in a host country. The study employed a research method where data were collected through online searches and analyzed thematically. The study found three major benefits. Firstly, APEC was seen to provide an opportunity for PNG to showcase a positive destination image to the world. Secondly, the infrastructures built for APEC will remain as legacies of mega-events and continue to benefit PNG. Thirdly, the study revealed that APEC in PNG helped the growth of the tourism and hospitality industry, which contributes to economic development. The study provides a better understanding of hosting a mega-event in a developing country.

Keywords: Asia Pacific Economic Cooperation, destination branding, Divine Word University, economic, hospitality, infrastructure, marketing, mega-event, Papua New Guinea, promotion, tourism.

Introduction

In recent years, mega-events have developed into one of the most important platforms for countries to showcase their customs, attractions and values on the international stage (Dean, 2014; Knott et al., 2013). For the host nation, it is crucial to assess both positive and negative impacts of the mega-event on the residents, the businesses and the social wellbeing of the people (Dean, 2014; Knott et al., 2013). Examples of mega-events include the Rugby League (World Cup, FIFA World Cup, and the Olympic Games

(Dean, 2014; Knott et al., 2013). These events directly involve thousands of people, including athletes, official delegates, volunteers, media personnel, spectators and indirectly an audience of billions through media exposure (Dean, 2014; Knott et al., 2013). Hosting a mega-event involves many benefits such as infrastructure, nation branding, economic development, the establishment of international partnerships and helping tourism growth (Cashman, 1999).

This paper aims to answer the following research questions: (1) what are the benefits of hosting a mega-event? (2) How do mega-events help improve a destination image? Thus, the paper will discuss the benefits of hosting a mega-event on a tourism destination and how mega-events help improve a destination image.

Mega-events and nation branding

Mega-events are global events that get global media attention, which is a powerful tool for reinventing and branding a country's image (Higham & Hinch, 2009). They provide a great opportunity to market and showcase a destination with some degree of associated benefits for the host nation (Pryke, 2017). The benefits include positive destination image, infrastructure development, and investment and business opportunities (Berkowitz et al., 2007; Higham & Hinch, 2009). They also have the potential to promote national unity and infrastructure development (Gratton & Preuss, 2008). For instance, when Australia hosted the Sydney Olympic Games in 2000, it generated massive news coverage internationally and increased tourism growth and infrastructure development (Florek & Insch, 2011).

Similarly, when Papua New Guinea (PNG) hosted the 2015 South Pacific Games in Port Moresby, the awareness of PNG as a tourism destination grew, which was particularly important since PNG receives poor negative media publicity overseas (Oxford

Business Group, 2018; Pryke, 2017). In preparation for the 2015 Pacific Games, the PNG government built an array of state of the art sporting and other facilities in Port Moresby. These include the Rita Flynn Complex, Taurama Aquatic Centre, Bisini Sports Venue, Sir John Guise Indoor Arena, Sir John Guise Outdoor Stadium, Games Village, Sir Hubert Murray Stadium, Oil Search Stadium and the Kumul Flyover (Kenneth, 2016; Oxford Business Group, 2018). In preparation for the Asia Pacific Economic Cooperation (APEC) Meeting, the PNG government and other donor agencies built world-class facilities such as the International Convention Centre and APEC House at Ela Beach in the shape of PNG's unique Lakatoi (traditional canoe) sail (Oxford Business Group, 2018; Pryke, 2017).

In 2018, new road infrastructure was built as part of the National Capital District's Road Rehabilitation Project comprising five bridges, a ring road and a road to the central meeting area of APEC House (Oxford Business Group, 2018). The private sector, especially the hotel industry, underwent major development, including the Hilton Hotel which has Kutubu Convention Center, the Holiday Inn Express and Stanley Hotel. These infrastructure projects were built for the South Pacific Games and APEC meeting but are still used today, being a legacy of mega-events (Oxford Business Group, 2018).

Media and destination image

Both positive and negative images of a tourist destination are reported through different mediums that help market a destination. According to Gartner (1993), these are called 'image formation agents' and relate to the mediums through which information is transmitted. The image formation process is a continuum comprising of different media that perform individually or combined to create a destination image appealing to tourists (Gartner, 1993; Khodadadi, 2013; Sonmez & Sirakaya, 2002). Gunn (1972) classifies two ways for the image of a tourist destination to be formed: organically (unintentional information transmission) and induced (image formed by active promotion). The main difference between organic and induced images is the absence of control by destination management organizations over the organically created destination image (Gartner 1993).

The concern for the PNG Tourism Promotion Authority (PNG TPA) is there is no control over media reporting in PNG. The PNG Government spent a substantial amount of money on tourism development. For instance, in 2019 the PNG government allocated PGK50m to promote and market PNG as a desirable tourism destination but there seems to be less or no control over media reporting in PNG that filters and regulates media reporting (Oxford Business Group, 2019). It can be argued that its part of media freedom but must be reported based on facts (Sumb, 2020). Consequently, PNG continuously receives a very poor destination image among international source markets (Sumb, 2020). For instance, almost every day PNG on the front page of the *Post Courier* or *The National* have reports of violent crime, rape, murder and other related social issues (Sumb, 2020). This also includes travel warning posted by Australia, New Zealand, United States and European countries. The reports are centred on safety and security issues in PNG (Sumb, 2020).

Different media organizations regard safety and security as important news generators. Thus, when an incident related to safety and security occurs, the media becomes avidly interested in reporting to its audience about the incident (Khodadadi, 2013; Pizam & Mansfeld, 2006; Sumb, 2019). This has a negative influence on potential tourists in the generating markets when the issue involves world media coverage, which creates a strong negative image among potential tourists (Khodadadi, 2013; Sumb, 2019; Tan et al., 2017).

The literature has shown that the higher the frequency of such incidents and the more media coverage it receives, the greater the negative impact on the tourist demand (Khodadadi, 2013; Lauderdale et al., 2011; Sumb, 2019). The high rate of security and safety incidents causes changes in tourists' booking and cancellation of tours (Pizam & Mansfeld, 2006; Sumb, 2019/2017). This can be measured within a given period and corresponds with the frequency patterns of daily crimes committed (Pizam &Mansfeld, 2006). The potential tourists translate this as unacceptably high risk and might cancel their bookings or choose to book alternative and more secure destinations. It is, therefore, in the interest of the tourism industry and host country to try to balance the negative images by conveying a wider range of feature stories of attractions and tourism activities (Pizam & Mansfeld, 2006; Sumb, 2019/2017).

Thus, one way to promote positive destination image is through hosting international mega-events that get worldwide media coverage (Sumb, 2020). For instance, PNG hosted several international events in Port Moresby such as the 2015 Pacific Games, the 2017 Rugby League World Cup and the 2018 APEC meeting (Kenneth, 2016). These events not only promote tourism but also provide several other benefits such as infrastructure and economic development (Pryke, 2017).

According to Cashman (1999), mega-sporting events help form a powerful destination branding despite a previous negative image. The benefit of hosting a mega-event creates a positive destination image. Since the event is broadcasted internationally and viewed by many people around the globe, it is likely to promote the host destination.

Methodology

This research used a qualitative method to answer questions about the benefits of APEC meeting in Port Moresby, PNG. The qualitative data were collected using desktop research such as searching websites, research papers and news articles to find evidence of benefits of APEC in PNG published between January 2016 and October 2020. These methods were used to collect data as all data were from secondary sources and online-based. Searches were done using the keywords “benefits of APEC in PNG”. The databases used include Wiley Online Library, Science Direct, JSTOR, Google Scholar and Research Gate. Searches were also made on organizational and governmental websites, such as those of PNG Tourism Promotion Authority (PNG TPA), Australian High Commission (PNG), Post Courier, The National and the Department of Pacific Affairs in Australian.

There were 22 papers found, which related to benefits of APEC in PNG. Of the 22 papers, 10 were from websites, 10 from research journals and the remaining two were news articles. Most of the papers were published in 2018, three in 2019, two in 2020 and one each in 2017 and 2016 respectively. Thus, not many scholarly academic articles were written about the benefits of APEC in PNG (Appendix 1).

The data were analyzed using the thematic analysis (Table 2) involving the following process:

1. 22 papers related to the research questions were reviewed and became familiar with the content.
2. An excel spreadsheet was created and all the titles of the papers, authors, year published and coding section were entered.
3. Codes were identified in all the papers
4. Themes were developed based on the codes
5. Themes were reviewed and started work on the scope
6. Finally wrote the findings and discussion (Knight, 2002).

Table 2. Coding of themes

Initial themes	**Finding coding themes**
Mega-events and nation branding	APEC promotes positive destination image
Image formation agents and mega-events	Infrastructure and legacies of APEC
	Economic benefits of APEC
	Security operations for APEC leaders

Findings and discussion

Positive destination image

It was found that there was evidence of a positive destination image for PNG. The country was on "global spotlight" when it hosted the APEC meeting (Australian High Commission – PNG, 2018). This placed PNG on the world map as global media attention was on PNG. It helped PNG showcase its culture, people and economy to the world (Wenogo, 2018). The APEC meeting was reported around the world and created a good image for the country (APEC, 2018; Business Advantage PNG, 2018). This was made possible by major media organizations from the "21 economics with 3500 media personals" (Post Courier, 2018). This includes China's State Media the China Global Television Network group, CCTV News, Russian media, Japan, Malaysia, Vietnam, Brunei Darussalam, Canada, Chile, People's Republic of China, Hong Kong, Indonesia, Japan, Republic of South Korea, Malaysia, Mexico, Peru, Republic of the Philippines, Russia and the United States of America (Post Courier, 2018).

The APEC was a historic event that promoted and marketed PNG in a single venue that was attended by media from around the world, who may have had perceptions that PNG is an unsafe tourism destination from rumours and stories (PNG TPA, 2018; Wenogo; 2018).The same sentiments shared by the Australian High Commission in PNG:

> "Leading Asia-Pacific Economic Cooperation (APEC) experts say Papua New Guinea stands to benefit from being in the global spotlight when it hosts the APEC forum in 2018, including an anticipated boost

> to investment, tourism and trade" (Australian High Commission-PNG, 2018, para. 1).

This contributed to the growth of the tourism and hospitality industry. The findings of this study are consistent with the literature on the benefits of the host nation in destination branding (Higham & Hinch, 2009).

Infrastructure and legacies

Most of the articles reported that in preparation for the APEC, the PNG Government and international donor partners such as China built several major infrastructures in Port Moresby (Kama, 2018; Kenneth, 2016; McLeod & Pryke, 2018; Oxford Business Group, 2018/19). These include the building of new and resealing existing roads, the building of the International Convention Centre and the APEC House.

The APEC also led to the building of the - Hilton Hotel and the Stanley Hotel (Kama, 2018; Jones, 2018; APEC, 2018; Oxford Business Group, 2018/19). China fully supported PNG to host the APEC meeting and funded major projects such as the redevelopment of PNG's International Conference Center and the building of a grand boulevard connecting it to Parliament House (Zhe, 2018). However, to balance Chinese's presence in PNG, the United States and its allies (Australia and Japan) used the APEC summit to announce their major projects to support PNG as highlighted by Oxford Business Group:

> "In an effort to balance Chinese investment, the US and its allies Australia and Japan used the APEC summit to unveil a major project of their own, announcing a $1.7bn plan to help PNG achieve its goal of supplying electricity to 70% of the population by 2030. Currently, less than 15% of the population, mainly those in urban areas, have access to a reliable supply of electricity, and this is seen as a major restriction to economic development" (Oxford Business Group, 2018, para. 10).

These findings on the value of infrastructure development for APEC are similar to that found in the literature (Berkowitz et al., 2007; Pryke, 2017). These are legacies of mega-events that remain for the long term in the host destination. The event is hosted for one or two weeks but the infrastructure remains and can be used for other future events. These are the direct benefits of hosting a mega-event (Cashman, 1999). For instance, as highlighted by Kama:

> "The APEC show began with new hotels erected, roads paved and the construction of the spectacular APEC House on Ela Beach" (Kama, 2018, para. 2).

Economic benefits of APEC meeting

The articles reported that there was evidence of economic activities during the APEC meeting (PNGTPA, 2020; Pryke, 2018). For instance, Port Moresby residents received monetary benefits through their engagement as volunteers and cleaners while companies received contracts to provide transportation, hospitality, catering and other services (Wenogo, 2018). Also, all the hotels in Port Moresby were fully booked and cruise ships were brought to provide extra beds. This was stated by Werner in conference proceedings in Australia:

> "Papua New Guinea hosted the APEC 2018 Summit in November 2018. Due to the shortfall of hotel rooms during the meeting peak times, the PNG Government decided to use three Cruise Ships, berthed in Fairfax Harbour, for providing the additional accommodation capacity"(Werner, 2019, pp. 437-443).

The meeting also aimed to raise awareness of the business opportunities and integrate the Asia-Pacific region economically by removing the trade barriers and keeping the spirit of trade, investments and business in the region (APEC, 2020; PNGTPA, 2020).

Security operations for APEC leaders

PNG is seen to have a negative destination image associated with safety and security issues (Kama, 2018; Lyons, 2018, Sumb, 2020/19/17; Wenogo, 2018). As discussed by Sumb in his paper titled Developing PNG's tourism sector:

> "The media has reported on all types of crime committed in the country, with incidents making headlines both locally and internationally. Thus, the perception of PNG overseas is that it is a dangerous and unsafe tourist destination" (Sumb, 2020, p. 8).

This was a concern for the PNG government, which needed to ensure that there were no or a few incidents during the APEC meeting. This was made possible by the Australia Government in providing security assistance including the following (APEC, 2018):

- Australian Defense Force provided security assistance across the maritime, aviation and counter-terrorism domains.
- Training and capacity building for PNG Defense force to develop skills and capabilities
- Australian Federal Police provide training and improve infrastructure for Royal PNG Constabulary.
- A cybersecurity package including cybersecurity controls in ICT infrastructure and a Cyber Security Operations Centre.

Also, there was a heavy presence of the military in the country not only from Australia but other countries as well to protect some of the world's most powerful leaders who were attending the APEC meeting. The justification for taking such a measure could be for the following reasons. First, the world is experiencing unprecedented terrorism activities and anything could happen when world leaders were in the country. Secondly, PNG was rated as an unsafe city in the world by foreign media such as The Guardian. This is the quote from The Guardian:

> "As the city, which is one of the most dangerous in the world, prepares itself for the arrival of an estimated 5,000-7,000 people including world leaders, CEOs and journalists, we asked our Papua

> New Guinean readers what they thought of their country, the poorest in the Apec bloc, hosting the meeting" (Lyons, 2018, para. 1).

With such negative media publicity, PNG managed to host 2018 APEC leader's meeting without any major incidents which promoted good destination image for PNG.

Conclusion

Mega-events have created a powerful destination image for a host nation with media coverage from all over the world. This paper presented the benefits of hosting APEC in PNG.

PNG benefited in three ways. Firstly, PNG was able to showcase a powerful destination brand, which resulted from a team effort between the private and public sectors. Secondly, PNG benefited from the infrastructure built by overseas partners and the PNG government. These are legacies of mega-events and will and continue to benefit the country. Thirdly, PNG experienced growth in the hotel industry, which could contribute to economic development and employment opportunities. APEC member countries such as Australia, United States of America, China, Japan, and New Zealand promised to support PNG in its development aspirations in business, trade and infrastructural developments.

Some foreign media feel that PNG might not be able to host APEC meeting successfully due to significant debts, the inability to maintain expensive new infrastructure and the diversion of scarce public funds away from priority areas, with benefits confined to the capital city. However, PNG was able to host APEC with these positive benefits as highlighted above in this paper. The findings of this study are based on a qualitative study and data collected through secondary sources. A future study of a similar event should include a quantitative method and primary data to build on from this study.

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Author

Allan Sumb is a lecturer within the Department of Tourism and Hospitality Management at DWU. He comes from Western Highlands Province, PNG. He holds a Master's Degree in Tourism from Otago University, New Zealand and is currently undertaking a Master of Research Methodology at DWU. Email: asumb@dwu.ac.pg.

Appendix

Summary of online sources

Author	**Year of publication**	**Title of article**	**Source**	**Comments**
PNG TPA	2020	APEC study reveals cruise tourism impacts in PNG	TPA Website	Generate revenue for PNG
Allan Sumb	2020	Developing Papua New Guinea's tourism sector	ANU Open Library - Research Paper	PNG's development challenges
Gebaue Werner	2019	Upgrade of Fairfax harbor berthing facilities for APEC 2018 Port Moresby, Papua New Guinea	Conference Paper	Cruise ships were brought into PNG to provide extra rooms for APEC visitors
Allan Sumb	2019	International visitors' perceptions of safety and security issues in Madang, PNG	South Pacific Studies - Research Paper	Poor destination image for PNG
Oxford Business Group	2019	Major construction projects in PNG ahead of APEC 2018	Oxford Business Group- Website	Major infrastructure development in PNG including APEC infrastructure
Shane McLeod & Jonathan Pryke	2018	2018 Australia–Papua New Guinea emerging leaders dialogue outcomes report	Lowy Institute- Research Report	Dialogue between PNG and Australia in preparation for APEC 2018
Bal Kama	2018	The 'year of APEC' – reflections on PNG in 2018	Devpolicy - Research Paper	People's expectations of APEC benefits & challenges
Kate Lyons	2018	Can they really pull it off?: the APEC summit comes to Papua New Guinea	The Guardian- News Report	Foreign media doubted about PNG's ability to host the APEC
Jonathan Pryke,	2018	Future scenarios for Papua New Guinea	Lowy Institute- Research Paper	Future development challenges for PNG if APEC not managed well

Jenny Hayward Jones	2018	Changing geopolitical dynamics for Papua New Guinea	Lowy Institute-Research Paper	Likely impact of PNG's hosting of the APEC summit in 2018
Australian High Commission, PNG	2018	APEC to put Papua New Guinea in the global spotlight	Australian High Commission- PNG Website	APEC promotes positive image for PNG
APEC	2018	Papua New Guinea's APEC turning point	APEC-Website	Opportunities and benefits of APEC
Jeremiah Wenogo	2018	For PNG's sake let's hope hosting APEC is for the better	Devpolicy Blog-Research Paper	Benefits of APEC in terms of infrastructure, employment and destination image
Oxford Business Group	2018	Papua New Guinea reaps benefits from hosting APEC summit	Oxford Business Group-Website	Benefits of APEC in PNG
Business Advantage PNG	2018	How the world's media reported on APEC 2018 in Papua New Guinea?	Business Advantage PNG - Website	Global media in PNG for the APEC
Post Courier	2018	Media on APEC in PNG	Post Courier- News Report	PNG was on the spotlight with 3500 journalists from all over the world covering APEC in PNG
Oxford Business Group	2018	A rising international profile and unique niche offering support tourism market in Papua New Guinea.	Oxford Business Group-Website	Hotel infrastructure development in PNG
PNG TPA	2018	PNGTPA sees APEC 2018 as an opportunity to promote tourism and improve economic growth	PNG TPA -Website	Opportunity for economic growth
APEC	2018	APEC promotes tourism in Papua New Guinea	APEC – Website	APEC promotes tourism in PNG
Gong Zhe	2018	Highways, schools and medical aid: How China is helping Papua New Guinea's development	CGTN-Website	China supports Infrastructure development
Allan Sumb	2017	New Zealand travellers' perceptions of safety and security issues in PNG	PNG Contemporary Studies. DWU Research Journal	Safety and security issues in PNG
Gorethy Kenneth	2016	2016 year end review	Post Courier – News Article	Major infrastructure developments in preparations for APEC

Some issues in solving non-linear polynomial equations

Peter K. Anderson

Abstract

A simple geometrical problem generates a degree 8 polynomial function after firstly applying Pythagoras' Theorem and then squaring the resulting equation to derive a more elegant polynomial equation. Real and complex solutions are explored by root finding functions available in R packages together with other readily available software. The degree 8 polynomial has 4 real and 4 complex solutions as expected. Squaring introduces extraneous solutions and only one of the final 8 solutions solves the Crossed Ladders problem explored in the text.

Keywords: similar triangles, polynomials, R, 1[D] root finding.

Introduction

Some mathematical problems are innately complicated, and some, perversely, need to be made so, according to the dictum of a great mathematician: "Mathematics makes easy things hard in order to make hard things easy"[1]. There are some, however, which are framed in simple terms, but which turn out to be unexpectedly difficult. From Archimedes (c. 500 BC) is said to have originated the famous "cattle problem" (Dickson, 1919). It involved numbers of cattle of different sorts and different colours and involved solving an equation with only two squares[2]. However, an integer solution consisted of 41 digits! The NQueens problem (Campbell, 1977) involving the positioning 8 queens on an 8 x 8

[1] TG Room, Advice to students, Professor of pure mathematics University of Sydney 1935-40, 1945-68.

[2] https://mathworld.wolfram.com/ArchimedesCattleProblem.html

chessboard and had 12 fundamental solutions[3], yet it took Gauss (1777 – 1855) 2 years to complete the solution.

In the light of such experiences, this paper will consider a seemingly simple geometrical problem, the Crossed Ladders Problem[4] (source unknown), which although hardly meriting the description of a 'problem', nonetheless, is not without its interest, generating, as it does, an 8 degree polynomial equation from a simple analysis using the well-known Pythagoras' Theorem. What might once have taken years to solve in detail, is now readily solvable with computer software.

Crossed ladders problem

We provide a context for solving polynomial equations with equations resulting from applying Pythagoras' theorem to solve the Crossed Ladders Problem. Consider two ladders of respective lengths 3m and 4m leaning across a narrow path by making angles with vertical walls on each side (Figure 1). Their crossing point is at a height of 1m above the path. We are required to find the width of the path.

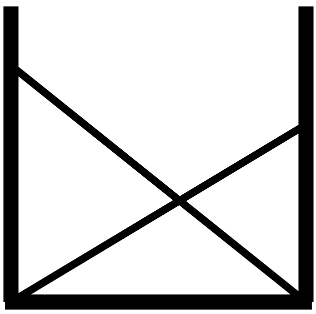

Figure 1 Two ladders of respective lengths 3m and 4m lean across a narrow path by making angles with vertical walls on each side.

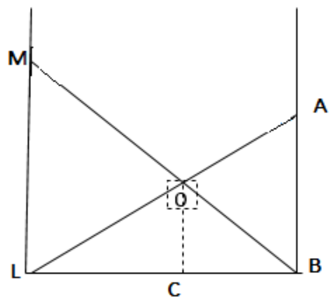

Figure 2 shows again the two ladders with a perpendicular line drawn down from their intersection point O.

[3] https://www.geeksforgeeks.org/n-queen-problem-backtracking-3/

[4] https://mathworld.wolfram.com/CrossedLaddersProblem.html

The corresponding labelled diagram (Figure 2) is shown with a perpendicular line drawn down from the intersection point O, set at 1m above the path. We let $A0 = x$, $B0 = y$, $AB = p$, $OL = 3 - x$, $OM = 4 - y$, $ML = q$, $BL = d$, and lastly, $BC = l$. We are required to find d.

Examination of the figure shows a number of relationships, the first stemming from Pythagoras' theorem:

$$l^2 + 1 = y^2 \tag{1}$$

$$d^2 + q^2 = 16 \tag{2}$$

$$\frac{1}{y} = \frac{q}{4}. \tag{3}$$

Then, combining (1), (2) and (3) we have:

$$\frac{1}{\sqrt{l^2+1}} = \frac{\sqrt{16-d^2}}{4} \tag{4}$$

Similarly, triangles LCO and LBA yield the further relationships:

$$(d-l)^2 + 1 = (3-x)^2 \tag{5}$$

$$d^2 + p^2 = 9 \tag{6}$$

$$\frac{1}{3-x} = \frac{p}{3} \tag{7}$$

Combining these three equations we have:

$$\frac{1}{\sqrt{(d-l)^2+1}} = \frac{\sqrt{9-d^2}}{3}. \tag{8}$$

Now equations (4) and (8) contain only l and d. Since the former is not needed, an equation in d, the required width, may be formed by making l the subject of each equation. Thus, after separately squaring (which will, of course, introduce unwanted solutions) and inverting each of equations (4) and (8), we obtain:

$$l = \sqrt{\frac{16}{16-d^2} - 1} \tag{9}$$

and so

$$l = d - \sqrt{\frac{d^2}{9-d^2}} \tag{10}$$

whence

$$\sqrt{\frac{d^2}{16-d^2}} = d - \sqrt{\frac{d^2}{9-d^2}}. \quad (11)$$

Eliminating *d* (presuming $d \neq 0$) from the numerators yields the following equation:

$$\sqrt{\frac{1}{16-d^2}} + \sqrt{\frac{1}{9-d^2}} - 1 = 0. \quad (12)$$

which, although not in a particularly elegant form for solving, at least, contains only *d*.

Initial inspection

The following section of the paper uses command lines from the R statistical computing and graphics programming language. Denoting equation (12) as a function *f*, graphing is obtained using the following two commands from the Curve function[5] in R:

(i) curve(f, from = -3, to = 3, col = “red”, lwd = 2, xname = “d”)

(ii) abline(h = 0, lty = 3).

The second command (ii) contains standard R graphical parameters; in the first command (i) the limits -3 and 3 come from the obvious physical constraints of the problem. Thus, initial plotting of

$$f12(d) = (1/(16-d^2))^{0.5} + (1/(9-d^2))^{0.5} - 1$$

(Figure 3) indicates 2 possible solutions to eqn. (12) here designated as *f12(d)* = 0. The negative value of d, of course, is not physically tenable and was introduced by the squaring process.

[5] https://www.rdocumentation.org/packages/graphics/versions/3.6.2/topics/curve

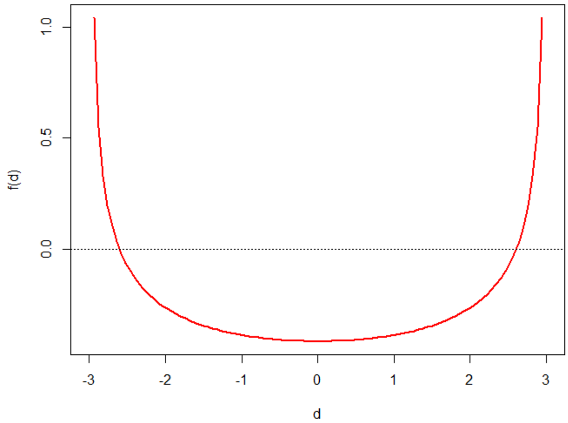

Figure 3 Initial plotting to search for solutions to equation (12) where *f12_(d)* = 0. Two solutions are evident with the negative value of d, of course, not physically tenable and introduced by the squaring process

Solutions from uniroot in R

A root-finding technique for polynomials within an interval is available with the function *uniroot* in the R *rootSolve* package. Based on the well-known bisection method of successive approximation, it requires the input of a function, and an interval, at one end of which the function is positive, at the other, negative and so passing through zero. This is a strong assumption, often calling for the inspection of the function by plotting as provided above (Figure 3).

Depending on the difficulty of the function, plotting may be required to find two points which may enclose a root. Then the R command would be of the following form:

uniroot(f, c(0,3)).

Since *f* is a simple function, other possible arguments for *uniroot* (dealing with tolerance and maximum number of iterations) need not, in this instance, be specified. Their default values are displayed in the output, the key parts of which are the value of the root, and the value of the function at the root, typically not exactly zero, but approximately so.

Thus, applied to equation (12) with a search domain of [0,3] uniroot readily gives d = 2.603288, which finishes the problem there, noting that, while d = −2.603288 may be also a mathematical solution, it is non - applicable for the physical situation, being a negative of distance.

Further development

With some further algebraic development, it is possible to produce an explicit expression for *d*, i.e. one that is free of radicals. Thus, rearranging equation (12) and squaring (again raising the possibility of the further addition of superfluous solutions) gives:

$$\frac{1}{16-d^2} = -2\sqrt{\frac{1}{9-d^2}} + \frac{1}{9-d^2} + 1 \qquad (13)$$

which, after squaring again, yields:

$$\left[\frac{-151+25d^2-d^4}{(16-d^2)(9-d^2)}\right]^2 = \frac{4}{9-d^2}. \qquad (14)$$

Then after multiplying out and re-arranging we have the equation:

$$d^8 - 46\, d^6 + 763\, d^4 - 5374\, d^2 + 13585 = 0. \quad (15)$$

This is the degree 8 polynomial mentioned previously in the introduction, and there is some interest to be had in determining its roots. Denoting eqn. (15) as function *g15(d)* and again applying *uniroot*, after some plotting, the function yields d = 2.60329, as previously, with a function value at the zero of -0.0031118, although no other root.

Initial inspection

Initial plotting of

$$g15(d) = d^8 - 46\, d^6 + 763\, d^4 - 5374\, d^2 + 13585$$

(Figure 4) indicates 8 possible solutions (4 real and 4 complex) to eqn. (15) located symmetrically about the y axis and shown in greater detail in graphical displays (Figures 5 & 6).

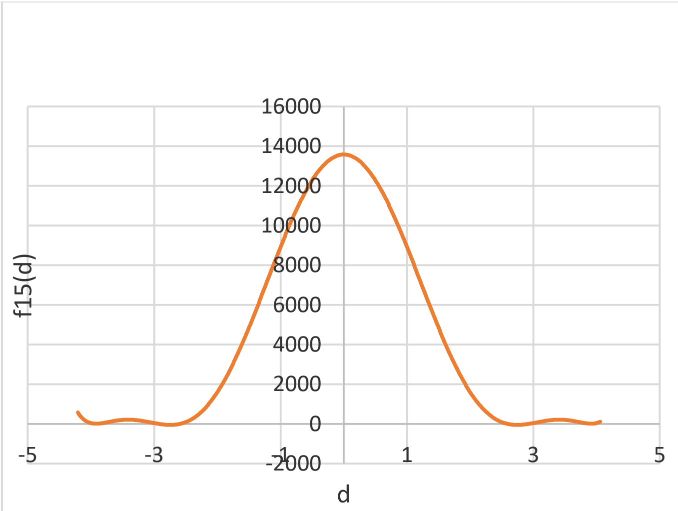

Figure 4 Initial graph of polynomial *g15(d)* permitting overall inspection.

There are 4 minimum turning points which would produce 8 real solutions if all minima lay below the x-axis and the curves cut the axis in 8 places. As only two minima lie above the x-axis, 4 out of the 8 possible solutions are complex.

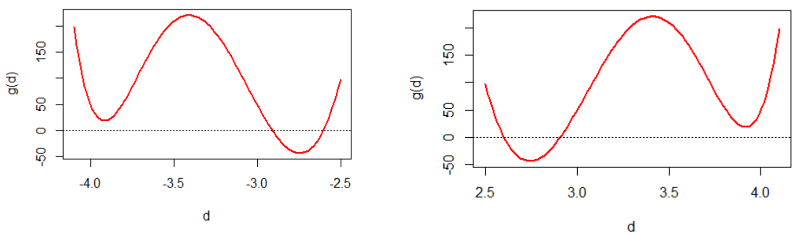

Figure 5 Plot of eqn. (15) LHS as a function of d locating approximate negative (LHS) and positive (RHS) positions of roots or real and complex zero solutions.

Solver optimisation tool

Solver, available as an Excel add-in, is an optimisation tool finding maximum, minimum, or zero values in an objective cell, subject to given constraints or limits[6]. With a given starting point,

[6] https://www.solver.com/excel-solver-online-help

Solver will stop after finding the first point satisfying the set conditions. Thus, it might find relative minima or maxima but not necessarily absolute values. The choice, therefore, has to be made of starting points for iterative searches after the overall shape of the polynomial curve has been determined (Figure 4).

Results of searches of polynomial *g15(d)* using the Solver optimisation tool (Table 1) show two zeros where a minimum lies below the x axis and a minimum only where it does not (Figure 5, RHS). In the latter case, we can expect to find complex roots. Initial starting points were determined by inspection of Figure 4. Solutions for –ve x starting points are not necessary because of the symmetry of the curve.

Table 1 Results of searches of polynomial *g15(d)* using the Solver optimisation tool showing two zeros where a minimum lies below the x axis and a minimum where it does not (Figure 5).

Initial value of d (Search starting point)	**Solution value of d**
2	2.60328781925415 (zero)
3	2.90907216330945 (zero)
4	3.91635300301455 (min)
5	3.91635300301455 (min)

Uniroot.all

Within R's rootSolve package there is also a function *uniroot.all* which is designed specifically to identify multiple function roots within an interval, by division into sub-intervals[7]. It needs to be made available first via the command *library(rootSolve)* (see Appendix). Then, the minimum syntax (omitting some optional arguments) is similar to that required for *uniroot*:

uniroot.all(g, c(2.90970, 2.90975)),

which provides a +ve root (Figure 5) as $d =$ 2.909072.

[7] https://www.rdocumentation.org/packages/rootSolve/versions/1.8.2.1/topics/uniroot.all

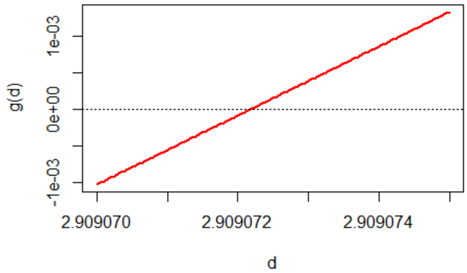

Figure 6 A closer examination again of shows the zero position between 2.909072 and 2.909073.

The syntax:

uniroot.all(g, c(2.0, 2.9)),

provides a second root (Figure 5) as d = 2.603288. Corresponding –ve roots are readily available by symmetry**.**

Polyroot

Also within R there is the *polynom* package[8], with a function **solve**[9] which is designed to reveal the complex roots of a polynomial. Here we enter the polynomial considered in this paper:

$$g15(d) = d^8 - 46\, d^6 + 763\, d^4 - 5374\, d^2 + 13585$$

in the following format:

```
polynomial(8:1)
8 + 7*x + 6*x^2 + 5*x^3 + 4*x^4 + 3*x^5 + 2*x^6 + x^7
> p <- as.polynomial(c(13585,0,-5374,0,763,0,-46,0,1))
> p13585 - 5374*x^2 + 763*x^4 - 46*x^6 + x^8
> solve(p)
```

[8] https://www.rdocumentation.org/packages/polynom/versions/1.4-0

[9] https://www.rdocumentation.org/packages/base/versions/3.6.2/topics/solve

Both real and complex solutions are obtained as follows (c.f. Figure 5):

(i) Real Solutions where *g15(d)* cuts the x axis:

-2.909072	-2.603288
2.603288	2.909072,

(ii) Complex solutions (Figure 7) where *g15(d)* minima lie above, and so does not cut, the x axis:

$z_1 = 3.922411 - 0.072176i$ $z_2 = 3.922411 + 0.072176i$

$z_3 = -3.922411 - 0.072176i$ $z_4 = -3.922411 + 0.072176i$.

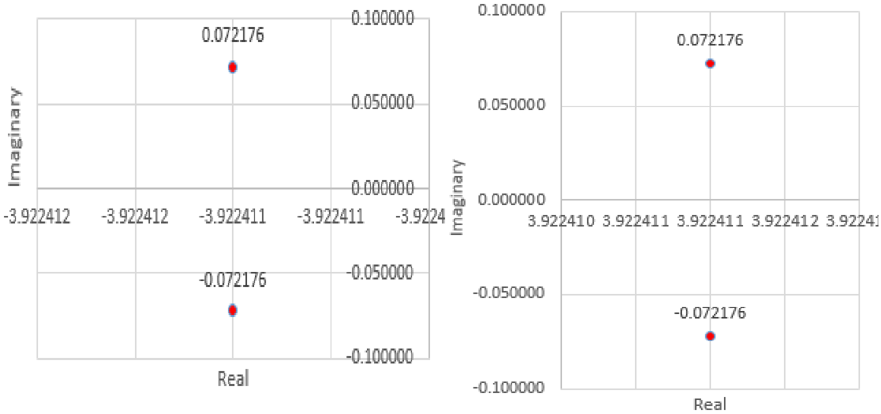

Figure 7 Complex roots (3.922411 ± 0.072176i, -3.922411 ± 0.072176i) shown on Real and Imaginary axes of an Argand diagrams to be compared with Figure 5 where minima do not cut the x axis and so complex or imaginary roots are formed.

These solutions are complex (imaginary) rather that real, but will still solve the g15(d) = 0 polynomial, with the imaginary components becoming real upon squaring.

Allroots( ) function in Maxima

Maxima is a computer algebra system used for the manipulation of symbolic and numerical expressions. It yields high precision numeric results and plots functions and data in two and three dimensions[10].

For solving again the polynomial:

$$g15(d) = d^8 - 46\, d^6 + 763\, d^4 - 5374\, d^2 + 13585,$$

the format used is:

```
 eqn: x^8-46*x^6+763*x^4-5374*x^2+13585;
              8          6              4                2
               x - 46 x + 763 x    - 5374 x  +  13585
soln: allroots (eqn);
```

giving the Real solutions (c.f. Figure 5):

x = 2.603287754423185, x = - 2.603287754423191,
x = - 2.909072186343945, x = 2.909072186343939.

and the Complex solutions (c.f. Argand diagram, Figure 7):

x = 0.07217649359882254 i + 3.92241066021693,
x = 3.92241066021693 - 0.07217649359882254 i,
x = 0.0721764935987751 i - 3.922410660216924,
x = - 0.0721764935987751 i - 3.922410660216924.

all of which satisfy *g15(d).* Thus, all 8 roots of the 8 degree polynomial are accounted for.

Summary and conclusion

A simple geometrical problem was shown to generate a degree 8 polynomial function after firstly applying Pythagoras' Theorem and then squaring the resulting equation to derive a more elegant polynomial equation. Real and complex solutions were found by root finding functions available in R packages together with other readily available software.

[10] https://swmath.org/software/560

The polynomial has 4 real and 4 complex solutions as expected. Squaring introduces extraneous solutions and only one of the final 8 solutions solves the Crossed Ladders problem. We notice that while the value d = ± 2.909072 satisfies equation (15), it does not satisfy equation (12), and so is not a solution to the original problem. There is, however, an interesting feature of this extraneous root: it is hidden away between 2.909070 where g15(d) is negative, and 2.909075 (Figure 6) where the function is positively making its existence virtually undetectable by plotting.

To summarize, there was an extra effort in the simple project of casting equation (12) into an elegant form, and in finding and discarding a spurious root, but there were also some useful learnings involved.

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Author

Peter K. Anderson, PhD

Head, Department of Mathematics & Computing Science, DWU

Email: panderson@dwu.ac.pg

Appendix – Code used

```
library(rootSolve)
#--------------------------------#
uniroot
#If equation (12) is described as f
        f <- function(d) { 1/sqrt(16 - d^2) + 1/sqrt(9 - d^2) -1 }
# and the search interval is 0 -> 3
        uniroot(f,c(0,3)) $root
# gives d = 2.603288 and for more information, enter:
        uniroot(f,c(0,3))
# showing that the function value at the root is of the order of 1e-
07.
# Thus:
        f(2.603288)
# satisfies equation (12), as by symmetry so does (-2.603288)
#--------------------------------#
The two following commands produce a plot of f:
        curve(f, from = 0, to = 3)
        abline(h = 0, lty = 3)
#--------------------------------#
if equation (15) is described as g
        g <- function(d){ d^8 -46*d^6 + 763* d^4 -5374*d^2 +
        13585}
# For uniroot, some experimentation with the search endpoints is
needed to produce opposite signs bracketing a root.
# Thus:
        uniroot(g,c(2.0, 2.9))$root
# gives d = 2.603288 as before.
#--------------------------------#

Applying uniroot.all using a similar syntax to uniroot:
        uniroot.all(g, c(1, 3))
# gives (i) 2.603291 which is a slightly less accurate value than
the previous 2.603288 because
```

```
    abs(g(2.603291)) > abs(g(2.603288))
# and also (ii) 2.909072, which, however, does not satisfy the original eqn (12) since f(2.909072) is not zero or approximately so.
# --------------------------------------------------#
```

Plotting the new root of (15)

```
    curve(g, from = 2.909070, to = 2.909075)
    abline(h = 0, lty = 3)
# --------------------------------------------------#
```

In Mathematica, g is defined as follows:

```
#  g[d_]: = d^8 -46*d^6 + 763* d^4 -5374*d^2 + 13585;
# The syntax for finding a root(s) of a function g of d would be:

# FindRoot[g, {d = s}] where 's' is some starting point.
# real and complex roots come from the one command
# ------------------------------------------------

# test the complex roots of g by substitution
    g(-3.92241 + 0.0721765i)
    g(-3.92241 - 0.0721765i)

    g(3.92241 + 0.0721765i)
    g(3.92241 - 0.0721765i)

# all of which give 0 (approximately).
```

Real & Complex solutions with Maxima

```
    eqn: x^8-46*x^6+763*x^4-5374*x^2+13585;
    8          6             4               2
    x  - 46 x  +    763 x  -    5374 x  +   13585

soln: allroots (eqn);
    x = 2.603287754423185, x = - 2.603287754423191,
    x = - 2.909072186343945, x = 2.909072186343939,
    x = 0.07217649359882254 i + 3.92241066021693,
```

x = 3.92241066021693 - 0.07217649359882254 i,
x = 0.0721764935987751 i - 3.922410660216924,
x = - 0.0721764935987751 i - 3.922410660216924

The result from *Maxima* was:

$$d = \pm(-3.92241 \pm 0.07217651i),$$
$$d = \pm\ 2.909072,\ \pm\ 2.60329,$$

Differentiability in normed spaces: A new approach

Raunu Gebo Sarsoruo

Abstract

The notions of limit, continuity, linearity and bilinearity are very substantial in the study of the general theory of differentiability in normed spaces. These concepts are used to provide precise proofs of differentiability of some functions in normed spaces. Common properties of the derivative of a function at a particular point are identified and expounded. The paper aims to show a new approach using common abstractive ideas to develop a better understanding of differentiation. Foundational concepts from limits that relate to continuity, then to linearity and bilinearity in the form of definitions, theorems and lemmas including some of their proofs provide a better way of understanding differentiability in calculus. Another significant result explored is the differentiability and continuity of implicit functions in Banach Spaces.

Keywords: Banach space, billinear, continuous linear mapping, continuous, converge, derivative, differentiable, implicit function limit, linear, norm, normed space, sequence.

Introduction

Interestingly, prevalent philosophical concepts that indicate the existence of continuity when a function is differentiable, are proposed to be a better way of understanding differentiation for university students and lecturers of calculus classes. Differentiation is a key concept in the study of calculus as a foundation for mathematical analysis. The main notion described is the differentiability of a function at a specific point. It seems to hold true for almost all occurring results from certain definitions, lemmas and the main theorems used. Each result seems to build upon the results from another theorem or lemma.

To achieve the main idea of this paper, which is to show that differentiability of a function at a specific point implies the existence of a limit and continuity at that same point, we begin from the notion of limits. It is known that the limit of a function f as x approaches a point $a \in \mathbb{R}$ can be found simply by calculating the value of the function f at point a. The concept of continuity builds on this property where functions are said to be continuous at a. If functions are continuous at a, then there exists a tangent line at that point. Such a line introduces the existence of a linear mapping at that point. If such a linear map exits at that point a, then we say that function f is differentiable at point a and satisfies the condition of differentiability stated as,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

This paper will also examine the differentiability of implicit functions and some observations are made when an implicit function is defined over a Banach space. Implicit differentiation is one of the many different techniques of differentiation which is useful for university students studying calculus.

Limit

The concepts of linearity, continuity and differentiability evolve firstly from the notion of limits. Limits are the foundational basis for continuity, linearity and differentiability to occur in normed spaces. Hence, the existence of a limit of a function at a point underlies the study of the theory of differentiability in general. Sequences are used to achieve a better and more concise understanding of limits.

We fix real linear spaces D, E over the same field $K \in \mathbb{R}$ and open set $V \subset D$ (Baron, 2019).

Definition 1 Function $f: \mathbb{Z}^+ \rightarrow \mathbb{R}$ given by $f(a_n)$ is called a sequence whose domain is a set of positive integers n.

Example 1 Let $f(a_n) = \frac{1}{n}$. The sequence whose n^{th} term is $\frac{1}{n}$ may be written as

$$f\{\frac{1}{n}\}_{n=1}^{\infty} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\}.$$

Remark 1 Observe that by choosing n sufficiently large, we find terms that are very close to zero. On the other hand, regardless of how large n is chosen, there are terms further out in the sequence that are not close to zero.

Example 2 Let $f(a_n) = \{\frac{(1+(-1)^n)}{2}\}_{n=1}^{\infty} = \{0,1,0,1,0,1, \cdots\}$. If we chose $n = 1{,}000{,}001$, then $a_{1,000,001} = 0$, but $a_{1,000,002} = 1$, which is not close to zero. If on a measure of closeness, say within $.025$ of zero, it is clear that all terms of the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ beyond the 40^{th} term satisfy this criterion. This idea leads to build the definition of convergence.

Definition 2 Function $f: \mathbb{Z}^+ \rightarrow \mathbb{R}$ given by a sequence $f(a_n)_{n=1}^{\infty}$ $converges$ to a real number x if and only if for every $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$ we have $|a_n - x| < \varepsilon$.

Remark 2 The choice of N depends on the choice of ε. The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to zero by intuition. If this conclusion is correct, then for $\varepsilon > 0$, there exists an N such that, for $n \geq N, |a_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$.

Example 3 If $\varepsilon = .035$, then, for $n \geq 51, |a_n - 0| = \frac{1}{n} \leq \frac{1}{51} < .035$. Thus, for $\varepsilon = .035, N = 51$ satisfies the conditions of definition 2. Now, we show that $f\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to zero (Gaughan, 1993, P51).

- Choose $\varepsilon > 0$.
- Let N be an integer larger than $\frac{1}{\varepsilon}$.
- If $n \geq N$, then we have $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$.
- This means that if $a_n = \frac{1}{n}$ and $A = 0$, then for $n \geq N, |a_n - A| = \frac{1}{n} < \varepsilon$.
- Therefore, the sequence converges to zero.

Definition 3 A sequence $\{a_n\}_n = 1^{\infty}$ is said to be $convergent$ iff there is a real number $x \in D$ such that $\{a_n\}_{n=1}^{\infty}$ converges to x. If $\{a_n\}_{n=1}^{\infty}$ is not convergent, it is said to be $divergent$.

Remark 3 The unique number to which a sequence converges to is called the $limit$ of the sequence.

Definition 4 Let x be an element of a real linear space D. A set $V \subset D$ is called the $neighborhood$ of point x, if there exists an open set $U \subset D$ such that $x \in U \subseteq V$.

In other words, a set V is the neighbourhood of the point x, if $x \in IntV$, where $IntV$ means the interior of the set V.

Definition 5 Let P be a subset of a real linear space D. A point $c \in P$ is called an $accumulation\, point$ of P if every neighbourhood of c contains at least one point of P different from c itself (Gaughan, 1993, P64).

Example 4 Given the set $G = \{\frac{1}{n}\}$ where n is a positive integer. This set is the range of the sequence $\{\frac{1}{n}\}_n^\infty$. Example 3 shows that $\{\frac{1}{n}\}_n^\infty$ converges to zero. Thus, every neighbourhood of 0 contains infinitely many terms of the sequence; and, since all terms of the sequence are distinct (that is, if $m \neq n$, then $a_m \neq a_n$), every neighbourhood of 0 contains infinitely many points of the set G. Therefore, 0 is an accumulation point of the set G.

Definition 6 Let $f: D \to \mathbb{R}$ and x_0 be an accumulation point of D. Then, function f has a limit L at x_0 iff for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x \in D$ if

$$0 < |x - x_0| < \delta$$

then

$|f(x) - L| < \varepsilon$ as illustrated in Figure 1.

Definition 7 Let $f: D \to \mathbb{R}$ be a function defined on some open interval that contains the number b, except possibly at b itself. Then, we say that the *limit of* $f(x)$ *as* x *approaches* b *is* L, and we write it as

$$\lim_{x \to b} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that if

$$0 < |x - b| < \delta$$

then

$|f(x) - L| < \varepsilon$ as illustrated in Figure 1 (Gaughan, 1993, P65).

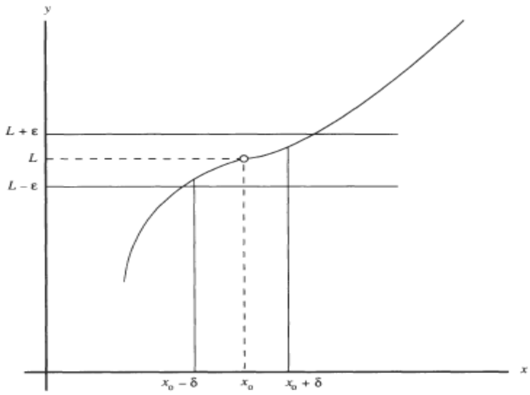

Figure 1: Illustration of $\varepsilon - \delta$ definition of limit (Gaughan, 1993, P65).

Example 5 Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ given by

$$f(x) = \frac{x^2-1}{x-1} \text{ for } x \neq 1, f(x) = \frac{x^2-1}{x-1} = x+1,$$

hence f is a linear function and the graph of f is a line with slope 1, except for $x = 1$. So, as x approaches 1, $f(x)$ approaches 2 filling in every necessary gap as possible. Hence, the limit of f at $x = 1$ is $L = 2$. Let us prove that f has a limit $L = 2$ at $x = 1$ using Figure 2.

Let us take $\varepsilon > 0$ as we consider the geometric interpretation of the idea of a limit outlined below.

- Choose a neighbourhood of 1 such that for x in this neighbourhood with $x \neq 1$.
- The corresponding points on the graph of f lie in the strip
$$\{(x, y): 2 - \varepsilon < y < 2 + \varepsilon)\}.$$
- Ignoring the point $x = 1$, the graph of f is a straight line of slope 1.
- Try $\varepsilon = \delta$ to obtain the neighbourhood $(1 - \delta, 1 + \delta)$ of $x = 1$.
- If $0 < |x - 1| < \delta = \varepsilon$, then

$$|f(x) - 2| = |\frac{x^2 - 1}{x - 1} - 2| = |(x + 1) - 2| = |x - 1| < \delta = \varepsilon$$

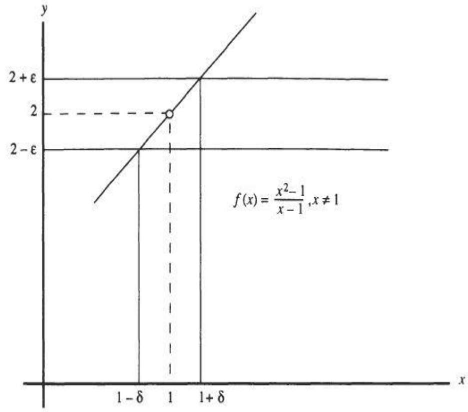

Figure 2: Example of $\varepsilon - \delta$ definition of limit (Gaughan, 1993, P66.)

Theorem 1 Let $f: D \to \mathbb{R}$ be a function with x_0 an accumulation point of D. Then, f has a limit at point x_0 if and only if for each sequence $\{x_n\}_{n=1}^{\infty}$ converging to x_0 with $x_n \in D$ and $x_n \neq x_0$ for all n, the sequence $\{f(x_n\}_{n=1}^{\infty}$ converges (Gaughan, 1993, P57).

Continuity

In the discussion of the limit of a function above, a function f has the property that if it has a limit at a point x_0, then it is said to be $continous$ at that point. Continuity in everyday language defines a process that takes place without interruption or abrupt change. Thus, a mathematical definition of continuity is closely related. Generally, we say that a function $f: D \to E$ is $continous$ at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

However, continuity can be defined more precisely using the definitions below that follows on from the notion of limits.

Definition 8 Suppose $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$. If $x_0 \in D$ then f is $continous$ at x_0 if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$|x - x_0| < \delta, x \in D$$

then

$$|f(x) - f(x_0)| < \varepsilon \text{ as displayed in Figure 3.}$$

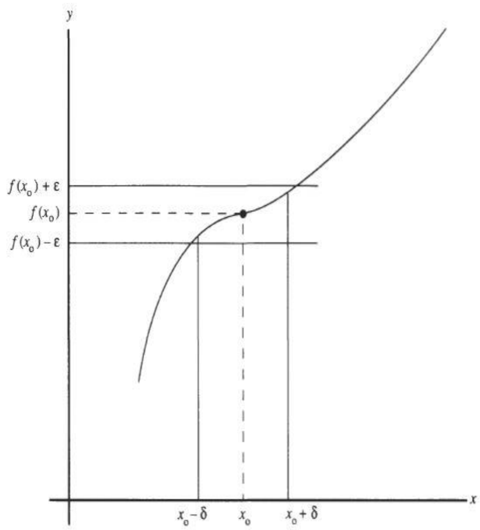

Figure 3: Illustration of $\varepsilon - \delta$ definition of continuity (Gaughan, 1993, P86.)

Remark 4 When comparing the definition (8) of continuity to the definition (6) of the limit of a function at point x_0, the following observations are made (Gaughan, 1993, P86).

- For continuity at x_0, the number x_0 must belong to D, but not an accumulation point of D.
- If $f: D \to \mathbb{R}$ with $x_0 \in D$ and x_0 not an accumulation point of D, then there is $\delta > 0$ such that if $|x - x_0| < \delta$ $x \in D$, then $x = x_0$.
- Hence $|f(x) - f(x_0)| = 0 < \varepsilon$ for every $\varepsilon > 0$.
- Now, we can say that if x_0 is not an accumulation point of D and $x_0 \in D$, then f is continuous at x_0 by default.
- Consider the case when x_0 is an accumulation point of D. Then, f has a limit at x_0 and that limit is $f(x_0)$.
- Comparing Figure 1 to Figure 3, f is continuous at x_0 if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that the graph of f for $x_0 - \delta < x < x_0 + \delta$, $x \in D$ lies in the strip $\{(x, y): f(x_0) - \varepsilon < y < f(x_0) + \varepsilon\}$.

Theorem 2 Let $f: D \to \mathbb{R}$ with $x_0 \in D$ and x_0 an accumulation point of D. Then the following conditions are equivalent (Gaughan, 1993, PP 87-88):

- For every sequence at $\{x_n\}_{n=1}^{\infty}$ converging to x_0 with $x_n \in D$ for each $n, \{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.
- f has a limit at x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$.
- f is continuous at x_0.

Proof

1. (i) $\Longrightarrow$ (ii)

   (a) Assume that (i) holds.

   (b) In particular, if $\{x_n\}_{n=1}^{\infty}$ converges to x_0 with $x_n \neq x_0$ and $x_n \in D$ for all n, then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.

   (c) Hence, by Theorem 1 f has a limit at x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$.

   (d) Thus (i) implies (ii).

2. (ii) $\Longrightarrow$ (iii).

   (a) Assume (ii) holds, and choose $\varepsilon > 0$.

   (b) Since (ii) holds, there is a $\delta > 0$ such that if $0 < |x - x_0| < \delta$ for $x \in D$, then $|f(x) - f(x_0)| < \varepsilon$.

   (c) If $0 < |x - x_0|$, then $|x - x_0| = 0$ implies that $x = x_0$.

   (d) Hence, $|f(x) - f(x_0)| = 0 < \varepsilon$.

   (e) Thus, f is continuous at x_0 and (ii) implies (iii).

3. (iii) $\Longrightarrow$ (i).

   (a) Suppose now that (iii) holds and that $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in D that converges to x_0.

   (b) Choose $\varepsilon > 0$. There is a $\delta > 0$ such that for $|x - x_0| < \delta$, for $x \in D$< $|f(x) - f(x_0)| < \varepsilon$.

   (c) Since $\{x_n\}_{n=1}^{\infty}$ converges to x_0, there is N such that for $n \geq N, |x_n - x_0| < \delta$.

   (d) Thus, for $n \geq N, |f(x_n) - f(x_0)| < \varepsilon$.

(e) This shows that $\{f(x_n)\}_{n=1}^{\infty}$. converges to $f(x_0)$.

Definition 9 Let a_i be a sequence of numbers for $a_i \in \mathbb{R}$. Sequence (a_i) is a Cauchy sequence, if

$$\underset{\varepsilon\in\mathbb{R}}{\forall} \quad \underset{\varepsilon\geq 0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{m,n>N}{\forall} \quad |a_m - a_n| < \varepsilon. \ (1)$$

This means that by selecting any small positive real number ε, a sufficiently large indicator N can be set such that any two expressions of higher orders are less than ε.

There are two equivalent definitions of real functions of a real variable. Let $M \subset \mathbb{R}$ and $f: M \to \mathbb{R}$.

Definition of Cauchy Function f is continuous in point $x_0 \in M$ if and only if:

$$\underset{\varepsilon\in\mathbb{R}}{\forall} \quad \underset{\delta\geq 0}{\exists} \quad \underset{x\in M}{\forall} \quad |x_0 - x| < \delta \to |f(x_0) - f(x)| < \varepsilon. \ (2)$$

The conditions

- $|x_0 - x| < \delta$ means that x belongs to the open sphere in the middle x_0 and radius δ.
- $|f(x_0) - f(x)| < \varepsilon$ means that $f(x)$ belongs to the open sphere in the middle $f(x_0)$ and radius ε.

Definition of Heine The function is continuous at point $x_0 \in M$, if and only if for each sequence (x_n) numbers from M, which is convergent to x_0, the string of values $(f(x_n))$ converges to $f(x_0)$, or

$$\underset{(x_n)\subset M}{\forall} x_n \to x_0 \to f(x_n) \to f(x_0). \qquad (2)$$

If the function f meets one of the above conditions for every $x \in M$, it is continuous on the set M respectively in the sense of Cauchy or the sense of Heine.

Linearity

We fix linear spaces W, X over the same field $K \in \{\mathbb{R}, \mathbb{C}\}$ and a set V (Baron, 2019).

Definition 10 A $linear\ (vector)\ space$ over the field K is called a set V with two binary operations defined as:

- addition of vectors: operation from the Cartesian product of the set V on set V, $+: V \times V \to V$, for vectors $u, v, w \in V$, we have $u = v + w$.
- scalar multiplication: operation from the Cartesian product of the set V and field K, $K \times K \to V$, for vectors $u, v \in V$ and number $a \in K$, we have $u = av$.

Definition 11 Let $f: W \to X$ be a mapping. We say that f is $linear$, if the following properties are satisfied;

- $f(u + v) = f(u) + f(v)$ for every elements, $u, v \in W$.
- $f(\lambda v) = \lambda f(v)$ for every $\lambda \in K$ and $v \in W$.
- $f(\lambda + \mu v) = \lambda f(u) + \mu f(v)$ for every $\lambda, \mu \in K$ and for every $u, v \in W$.
- $f(\lambda_1 v_1 + \cdots + \lambda_k v_k) = \lambda f(v_1) + \cdots + \lambda_k f(v_k)$ for every scalar $\lambda_1 \cdots \lambda_k \in K$, elements $v_1 \cdots v_k \in W$ and every $k \in \mathbb{N}$.

Property $f(u + v) = f(u) + f(v)$ is called the $additivity$ and property $f(\lambda v) = \lambda f(v)$ is called the $homogeneity$ of mapping f respectively.

Definition 12 Function $\|\cdot\|: X \to \mathbb{R}$ is called a norm in X, if for all $x, y \in X$, $\alpha \in \mathbb{K}$ the following properties hold:

- $\| x \| = 0 \to x = 0$
- $\| \lambda x \| = |\lambda| \| x \|$
- $\| x + y \| \leq \| x \| + \| y \|$

If $\|\cdot\|$ is a norm on X, then the pair $(X, \|\cdot\|)$ is called $normed\ space$.

Remark 5 If $\|\cdot\|$ is a norm in X, then for all $x, y \in X$ we have:

- $x = 0 \rightarrow \| x \| = 0$
- $\| x \| \geq 0$
- $| \| x \| - \| y| \leq \| x - y \|$ $\quad (0 = \| 0 \| = \| x + -x \| \leq \| x \| + \| -x \| = 2 \| x \|)$

Definition 13 Let X denote any non-empty set. A $metric$ on set X is called function $d: X \times X \rightarrow [0, +\infty)$ which for any elements $x, y, z \in X$ satisfy the conditions:

- $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

If d is a metric in set X, then the pair (X, d) is called $metric\ space$. Elements of set X are called $points$. Number $d(x, y)$ is called the $distance$ of a point x from point y.

Remark 6 If $\|\cdot\|$ is a norm on X, then function $d: X \times X \rightarrow [0, \infty)$ given by the formula $d(x, y) = \| x - y \|$ is a $metric$ on X.

Definition 14 Normed space $(X, \|\cdot\|)$ is called $Banach\ space$ if the metric space (X, d) is $complete$.

Remark 7 The completeness of the metric means that each $Cauchy\ sequence$ of elements in space X is convergent to some element of space X.

Differentiability in Normed Spaces

Let us fix normed spaces X, Y over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and open set $U \subset X$ (Baron, 2019).

Definition 15 Function $f: U \to Y$ is called differentiable in point $x_0 \in U$ if and only if when there exists such a continuous linear mapping $\Lambda: X \to Y$, that

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - \Lambda h}{\|h\|} = 0. \qquad (3)$$

Function $f: U \to Y$ is called $differentiable$ if and only if, when it is differentiable in every point of set U.

Remark 8 If function $f: U \to Y$ is differentiable in point $x_0 \in U$, then there exists exactly $one\ continous\ linear\ mapping$ $\Lambda: X \to Y$ satisfying condition (4). The continuity in this mapping means that:

$$\underset{\varepsilon>0}{\forall} \ \underset{\delta>0}{\exists} \ \underset{h \in X}{\forall} \ \left(0 < \| h \| < \delta \Longrightarrow \left\| \frac{f(x+h) - f(x) - \Lambda h}{\|h\|} \right\| < \varepsilon\right). \qquad (5)$$

- Assume that $\Lambda_1, \Lambda_2: X \to Y$ are linear mappings and $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - \Lambda_k h}{\|h\|} = 0$, for $k \in \{1,2\}$
- $\lim_{h \to 0} \frac{\Lambda_1 h - \Lambda_2 h}{\|h\|} = \lim_{h \to 0} \left(\frac{f(x_0+h) - f(x_0) - \Lambda_2 h}{\|h\|} - \frac{f(x_0+h) - f(x_0) - \Lambda_1 h}{\|h\|}\right) = 0.$
- Take $\varepsilon > 0$.
- $\delta \in (0, \infty): \forall_{h \in X} \quad (0 < \| h \| < \delta \Longrightarrow (\| \frac{\Lambda_1 h - \Lambda_2 h}{\|h\|} \| \leq \varepsilon)$.
- If $h \in X\{0\}$, then $0 < \| \delta \frac{h}{\|h\|} \| = \delta$ and therefore $\| \frac{(\Lambda_1 - \Lambda_2)\frac{\delta h}{\|h\|}}{\|\frac{\delta h}{\|h\|}\|} \| \leq \varepsilon$ that is $\| (\Lambda_1 - \Lambda_2) h \| \leq \delta \| h \|$
- $(\Lambda_1 - \Lambda_2) h = 0$ for $h \in X$
- $\Lambda_1 = \Lambda_2$

Example 6 Show that if $-\infty < a < b < \infty$, then function $f: C([a, b]) \to \mathbb{R}$ defined by the formula:

$$f(x) = \int_a^b \alpha(t) x(t)^2 \ dt$$

is differentiable and

$$f'(x)h = 2\int_a^b \varphi(t)x(t)h(t)dt$$

for $x, h \in C([a,b])$.

First, we check if the derivative of function $f(x)$ satisfies the notion of linearity and continuity. We fix

$$x \in C([a,b])$$

and define the function

$$\Lambda: C([a,b]) \to \mathbb{R}$$

by the formula

$$\Lambda h = \int_a^b \varphi(t)x(t)^2 \ dt$$

- $Linearity$:

  We fix elements $h, k \in C([a,b])$ and $\alpha \in \mathbb{C}$

- $Additivity$:

$$\Lambda(h+k) = 2\int_a^b \varphi(t)x(t)h(t) + k(t)dt$$

$$= 2\int_a^b \varphi(t)x(t)h(t) + \varphi(t)x(t)k(t)dt$$

$$= 2(\int_a^b \varphi(t)x(t)h(t)dt + \int_a^b \varphi(t)x(t)k(t)dt)$$

$$= 2\int_a^b \varphi(t)x(t)h(t)dt + 2\int_a^b \varphi(t)x(t)k(t)dt$$

$$= \Lambda(h) + \Lambda(k)$$

- $Homogenity$:

$$\Lambda(\alpha h) = 2\int_a^b \varphi(t)x(t)\alpha h(t)dt =$$

$$\alpha \cdot 2\int_a^b \varphi(t)x(t)k(t)dt = \alpha\Lambda(h)$$

- $Continuity$:

  We know that, $\| h \| = sup_{t\in[a,b]}|h(t)|$ for $h \in C([a,b])$.

$$|\Lambda h| = 2|\int_a^b \varphi(t)x(t)h(t)dt| \leq$$
$$\leq 2|\int_a^b |\varphi(t)x(t)h(t)dt| =$$
$$= 2|\int_a^b |\varphi(t)| \cdot |x(t)| \cdot |h(t)|dt| \leq$$
$$\leq \| h \| \underbrace{2\int_a^b |\varphi(t)| \cdot |x(t)|dt}_{\text{constant based on h}}$$

- $Differentiability$:

$$\lim_{h\to 0} \frac{|f(x_0+h)-f(x_0)-\Lambda h|}{\|h\|} =$$
$$\frac{|\int_a^b \varphi(t)(x+h)(t)^2dt - \int_a^b \varphi(t)x(t)^2dt - 2\int_a^b \varphi(t)xh(t)dt|}{\|h\|}$$
$$= \frac{|\int_a^b \varphi(t)([(x(t)+h(t))(t)]^2 - x(t)^2 - 2xh(t))dt|}{\|h\|}$$
$$=$$
$$\frac{|\int_a^b \varphi(t)x(t)^2 + 2\varphi(t)x(t)h(t) + \varphi(t)h(t)^2 - \varphi(t)x(t)^2 - 2\varphi(t)x(t)h(t))dt|}{\|h\|}$$
$$= \frac{|\int_a^b \varphi(t)(h(t)^2)dt|}{\|h\|}$$
$$\leq \frac{\int_a^b |\varphi(t)| + (|h(t)^2| \leq \|h\|^2|)dt}{\|h\|}$$
$$\leq \frac{\|h\|^2 \int_a^b |\varphi(t)|dt}{\|h\|} = \| h \| \int_a^b |\varphi(t)|dt \Rightarrow^{h\to 0} 0$$

(Baron, 2019).

Definition 16 If function $f: U \to Y$ is differentiable in point $x_0 \in U$, then one continuous linear mapping $\Lambda: X \to Y$ satisfying condition (4) is called $derivative\ of\ function\ f$ at point x_0 and is denoted by the symbol $f'(x_0)$ and so $f'(x_0)$ is a $continous\ linear\ mapping$ of space X in space Y and

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{\| h \|} = 0.$$

***Remark* 9** The continuity in the mapping Λ means that;

$$\underset{\varepsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{h\in X}{\forall} \quad \left(0 < \| h \| < \delta \Longrightarrow \| \frac{f(x_0+h)-f(x_)-f\prime(x_0)h}{\|h\|} \| < \varepsilon\right). \quad (4)$$

The following observations can be made:

- If $\Lambda: X \to Y$ is a continuous linear mapping, then Λ is a differentiable function and $\Lambda'(x) = \Lambda$ for $x \in X$.
- If X_1, X_2 are normed spaces and $\Lambda: X_1 \times X_2 \to Y$ are continuous billinear mapping, then Λ is differentiable and $\Lambda'(x_1, x_2)(h_1, h_2) = \Lambda(x_1, h_2) + \Lambda(h_1, x_2)$ for $(x_1, x_2), (h_1, h_2) \in X_1 \times X_2$.
- Homogeneity in each variable is bilinear such that we have $\Lambda(\alpha_1 x_1, \alpha_2 x_2) = \alpha_1 \Lambda(x_1, \alpha_2 x_2) + \alpha_2 \Lambda(x_1, x_2)$ (Baron, 2019).

Lemma 1 If X_1 and X_2 are normed spaces, then bilinear mapping $\Lambda: X_1 \times X_2 \to Y$ is continuous, if and only if, there exists such a constant $c \in (0, \infty)$, that

$$\| (x_1, x_2) \| \leq c \| x_1 \| \| x_2 \|$$

$$\text{for } (x_1, x_2) \in X_1 \times X_2.$$

Remark 10 If a function $f: U \to Y$ is differentiable in point $x_o \in U$, then function $u: U \to Y$ defined by the formula:

$$u(x) = \begin{cases} \frac{f(x)-f(x_0)-f\prime(x_0)(x-x_0)}{\|x-x_0\|} & \text{if } x \neq x_0, \\ 0 & \text{if } x = x_0, \end{cases} \quad (7)$$

is continuous in point x_0 (Baron, 2019).

Theorem 3 A function that is differentiable at a point is continuous at that same point.

Proof

- Assume that $f: U \to Y$ is differentiable in point $x_0 \in U$.
- Function $u: U \to Y$ defined by formula (7) is continuous in point x_0 and $f'(x) = f(x_0) + f'(x_0)(x - x_0) + \| x - x_0 \| u(x)$ for $x \in U$.
- $\lim_{x \to x_0} f(x) = f(x_0)$ (Baron, 2019).

Definition 17 Isom $(X, Y) = \{\Lambda: X \to Y | \Lambda$ is a one-to-one continuous linear mapping, $\Lambda(X) = Y$ and Λ^{-1} is a linear mapping.$\}$

Definition 18 Let $L(X, Y)$ denote all the family of linear and continuous operators in normed space X with respect to the values in normed space Y.

Definition 19 Function $f: U \to Y$ is called a C^1 class function if and only if, f is differentiable and $f': U \to L(X, Y)$ is continuous.

Remark 11 Continuity in the mapping, $f': U \to (X, Y)$ means that:

$$\underset{x_0 \in U}{\forall} \ \underset{\varepsilon > 0}{\forall} \ \underset{\delta > 0}{\exists} \ \underset{x \in U}{\forall} \ (0 < \| x - x_0 \| < \delta \Longrightarrow \| f'(x) - f'(x_0) \| < \varepsilon). \quad (5)$$

Differentiability of implicit functions in Banach Space

We begin this section by expounding the concept of differentiability to functions that are impossible to differentiate directly. Observe that a function of the form $y = f(x)$ is simple to differentiate. However, when it is inconvenient to express functions of this form, we tend to use functions that are defined implicitly. For instance, consider the function

$$yx + y + 1 = x.$$

This function is not expressed in the form of $y = f(x)$, but it still defines y as a function of x since it can be rewritten as

$$y = \frac{x-1}{x+1} \text{ (Anton et al, 2012, P185).}$$

Example 7 Consider the function $f(x) = 2y + x + 8 = 1$. Applying implicit differentiation on function f with respect to x results in;

$$\frac{d}{dx}(2y) + \frac{d}{dx}(x) + \frac{d}{dx}(8) = \frac{d}{dx}(1)$$
$$2\frac{dy}{dx} + 1 + 0 = 0$$
$$\frac{dy}{dx} = \frac{-1}{2}$$

We fix Banach spaces X, Y, Z and open set $D \subset X \times Y$ (Baron, 2019).

Theorem 4 If $f: D \to Z$ is a C^1 class function, $(x_0, y_0) \in D, f'_Y(x_0, y_0) \in Isom(Y, Z)$ and $f(x_0, y_0) = 0$, then there exists a neighbourhood $U_0 \subset X$ of point x_0, a neighbourhood $W_0 \subset Y$ of point y_0 and such a function $\varphi: U_0 \to Y$ of C^1 class function, that $U_0 \times W_0 \subset D, \varphi(U_0) \subset W_0, \underset{x,y \in U_0 \times W_0}{\forall} (f(x, y) = 0 \Leftrightarrow y = \varphi(x))$

Identified from the result of theorem (4), the following observations can be made.

- For all neighbourhoods of point $x, y \in U_0 \times W_0$, some function, $f(x, y) = 0$ if and only if $y = \varphi(x)$.
- The function $y = \varphi(x)$ is a C^1 class function which is differentiable and continuous as defined in definition (19).
- If $x, y \in U_0 \times W_0$, then

$$f(x, y) = 0 \Leftrightarrow F(x, y) = (x, 0)$$
$$\Leftrightarrow F|_{U_0 \times W_0}(x, y) = (x, 0).$$

Example 8 If X, Y are Banach spaces, $D \subset X \times Y$ is an open set, $(x_0, 0) \in D$, and $g: (x_0) = 0$, then for every available number close/near 0, denoted by α, there exist a neighbourhood $U \in X$ of point x_0 and such a function $\varphi: U \to Y$ of class c^m, that $\varphi(x_0) = 0$ and $(x, \varphi(x)) \in D$ and $\varphi(x) = \alpha g(x, \varphi(x))$ for $x \in U$ (Baron, 2019).

Theorem 5 We assume that $f: D \to Z$ is a C^1 class function and $\forall_{x,y \in D}(f'_Y(x, y) \in Isom(Y, Z))$. If $U \in X$ is an open set and $\varphi: U \to Y$ is such a continuous function that

$$\underset{x \in U}{\forall} x, \varphi(x)) \in D \wedge f(x, \varphi(x)) = 0),$$

then φ is a C^1 class function and

$$\varphi'(x) = -f'_y(x, \varphi(x)^{-1} \circ f'_x(x, \varphi(x))$$

for $x \in U$.

Example 9 If $U \in \mathbb{R}^N$ is an open set, $a \in U, f: U \to \mathbb{R}$ is a C^1 class function, $f(a) = 0$ and $f'(a) \neq 0$, then set

$$\{x \in \mathbb{R}^N: \sum_{j=1}^{N} f|_j(a)(x_j - a_j) = 0\}$$

is a tangent plane to set $f^{-1}(\{0\})$ in point a (Baron, 2019).

Conclusion

Limit and continuity indeed provide a very fundamental approach to understanding differentiability in mathematical spaces. It is evident from the results of the definitions and theorems that differentiability of a function in a normed space heavily relies on the existence of a limit and continuity at a specific point. Hence, a function that is differentiable at a particular point implies that the function is continuous and has a limit at that same point. We also observe that a C^1 class function defined implicitly over Banach spaces implies differentiability and continuity of some neighbourhood points.

Glossary

C^1- A function that is differentiable and continuous.
ℝ - A space of real numbers.
ℕ - Natural numbers.
$L(X, Y)$- Family of linear and continuous operators.
f- a function.
f' - a differentiable function.
$f: X \to Y$- a mapping.
$V \subset D$ - an open subset.

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Author

Raunu Gebo Sarsoruo is a lecturer in the Department of Mathematics and Computing Science at DWU. Email: rgebo@dwu.ac.pg

Random variables and convex functions in Stochastic orderings with its applications in mathematics

Cyril Sarsoruo

Abstract

Randomness is a phenomenon that is mathematically studied in probability theory. Random quantities obtained have to be distributed over some graphs for interpretation. Mean is a descriptive quantity in statistics and is studied well using the convexity theory in pure mathematics. The paper shows the basic relationship of probability to random variables followed with convex function. Merging probability and convexity notions, we obtain sufficient conditions in the Ohlin lemma for stochastic convex orderings. The main results of the paper show formulations of two theorems using the Ohlin lemma in their proofs.

Keywords: convex function, convex stochastic ordering, expectation, Hermite-Hadamard inequality, Jensen functional inequality mean, Ohlin lemma, random variables.

Introduction

Certain problems can be solved mathematically by translating a particular problem into its mathematical model. Developing a mathematical model requires knowledge of mathematical theories. The main purpose of this research paper is to show a process of establishing elementary mathematical definitions, theorems and intertwining them to produce complex useful lemmas or theorems. Here, the Ohlin lemma is an example of this process and it is derived from combining continuous random variables and convex functions. The Ohlin lemma is used as a tool to prove two useful theorems of Hermite-Hadamanrd and Jensen functional inequalities (Rajba, 2014). We begin the paper by stating the relationship of probability to random variables and then to its expected or mean value. Following a definition of a convex function and then intertwining it with continuous random variables, we obtain the Ohlin lemma with its sufficient conditions for stochastics convex ordering. The conditions of the Ohlin lemma are used to show simple proofs of Hermite-Hadamand and the general case of finite Jensen functional inequalities in the application part of the paper.

Probability and random variables

The purpose of probability theory is to model random experiments so that we can draw inferences about them (Rosen, 2012). An experiment is a procedure that yields one of a given set of possible outcomes. The sample space of the experiment is the set of possible outcomes. An event is a subset of the sample space. Laplace's definition of the probability of an event with finitely many possible outcomes will now be stated.

Definition 1 If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the probability of E is $p(E) = \frac{|E|}{|S|}$.

According to Laplace's definition, the probability of an event is between 0 and 1. Note that if E is an event from a finite sample space S, then $0 \leq |E| \leq |S|$. Thus,

$$0 \leq p(E) = \frac{|E|}{|S|} \leq 1.$$

Many problems are concerned with the numerical value associated with the outcome of an experiment. For instance, we may be interested in the total number of one bits in a randomly generated string of 10 bits or in the number of times a tail comes up when a coin is flipped 15 times. To study problems of this type we introduce the concept of a random variable.

Definition 2 Let S be a sample space and a function $X: S \rightarrow X(S)$ is said to be a random variable where $X(S) \subset R$.

From Definition 2, notice that a random variable is a function which assigns each possible outcome in the sample space to a real number. Example 1 gives an illustration of an application of definition 2.

Example 1 A fair coin is flipped 3 times. Let S be the sample space of 8 possible outcomes, and let X be a random variable that assigns to an outcome the number of heads in this outcome.

Random variable X is a function X: S → X(S) where X(S) = {0, 1, 2, 3} is the range which is the number of heads and
S={(TTT), (TTH), (THH), (HTT), (HHT), (HHH), (THT) ,(HTH)}

$$X(HHH) = 3,$$
$$X(HHT) = X(HTH) = X(THH) = 2,$$
$$X(TTH) = X(THT) = X(HTT) = 1,$$
$$X(TTT) = 0.$$

Combining Definitions 1 and 2, we arrive at the notion of the probability distribution of the random variable X forming Definition 3.

Definition 3 The probability distribution of a random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$, where $p(X = r)$ is the probability that X takes the value r.

The set of pairs in this distribution is determined by the probabilities p(X = r) for $r \in X(S)$. The probability distribution of random variable X from Example 1 is given by p(X=3) = 1/8, p(X=2) = 3/8, p(X=1) = 3/8, p(X=0) = 1/8.

A random variable X can be either discrete or continuous. A random variable X is discrete if it has a finite or countable number of possible outcomes that can be listed as in Example 1. A random variable X is continuous if it has an uncountable number of possible outcomes, represented by the intervals. This research concerns continuous random variables so we introduce the following definitions.

Definition 4 Let X be a continuous random variable. Then a probability distribution or probability density function *(pdf) of X* is a function *f (x)* such that for any two numbers a and b,

$$P\ (a \leq X \leq b) = \int_{a}^{b} f(x)dx.$$

Definition 5 Let *F(b)* be the cumulative distribution function, for a continuous random variable *X* that is defined for every number $b\ \in R\ \ by$

$$F(b) = P\ (X \leq b) = \int_{-\infty}^{b} f(x)dx.$$

We now define the average or the mean of the continuous random variables, which is called expectation, denoted *E(X)* that will be used in the study.

Definition 6 The expected or mean value of a continuous random variable *X* with *pdf f (x)* is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$

Generally, the expectations of the continuous random variable *X* can be represented and analyzed mathematically using convex functions.

Convex function

At the core of the notion of convexity is the comparison of means (Niculescu & Persson, 2006). In this research, we look at the expectations of continuous random variables in convex functions. A convex function is defined as follows.

Definition 7 Let $J \subset \mathrm{R}$ be an open interval. The function f: $J \to \mathrm{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \qquad (1)$$

for all $x, y \in J$ and $\lambda \in [0,1]$.

It is said to be strictly convex if the inequality (1) holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.

Definition 8 Let $J \subset \mathrm{R}$ be an open interval. A function f: $J \to \mathrm{R}$ is said to be Jensen convex if and only if it satisfies the Jensen functional inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \qquad (2)$$

for all $x, y \in J$.

If the inequality in (2) for $x \neq y$ is sharp then f is said to be *strictly Jensen convex*.

Theorem 1 Let $J \subset \mathrm{R}$ be an open interval, and let $f : J \to \mathrm{R}$ be a given convex function. The function f satisfies (2) and is continuous if and only if it satisfies inequality (1) for all $x, y \in J$ and every $\lambda \in [0,1]$.

The proof to theorem 1 is omitted, as it will not be used in this paper.

Ohlin lemma

Definitions 1 to 8 and Theorem 1 give, the mathematical bases to formulate the Ohlin lemma (Ohlin, 1969) which provides sufficient condition for stochastic convex ordering (5) in lemma 1.

Lemma 1 Let X and Y be two continuous random variables with finite expectations such that

$$E(X) = E(Y). \qquad (3)$$

*I*f their cumulative distribution functions F_X and F_Y cross exactly one time, i.e., for there exist a point t_0 such that

$$F_X(t) \leq F_Y(t) \text{ } if \text{ } t < t_0 \text{ } and \text{ } F_X(t) \geq F_Y(t) \text{ } if \text{ } t > t_0 \qquad (4)$$

then

$$Ef(X) \leq Ef(Y) \qquad (5)$$

for all convex functions $f : \mathrm{R} \rightarrow \mathrm{R}$.

The proof to lemma 1 is omitted because it involves theorems and other mathematical theories, which are not included in this paper and are not relevant in the context of this research.

Applications

The Ohlin lemma or lemma 1 is used as a tool, which is applied to the proof of the following theorems mainly used in convex stochastic convex ordering to compare the Means of the probability distributions of continuous variables. Firstly, the Ohlin lemma is used to give a simple proof of the Hermite-Hadamard inequality. This inequality gives us an estimate of the integral mean value of a continuous convex function (Rajba, 2014). Secondly, we introduce the general finite case of Jensen inequality and use the Ohlin lemma to prove it. The Jensen inequality in its simplest form states that the convex transformation of a mean is less than or equal to the mean applied after convex transformation (Kuzma, 1985). We formulate two theorems and apply conditions of Ohlin lemma to prove them.

Theorem 2 Let $J \subset \mathrm{R}$ be an open interval, and $f : J \rightarrow \mathrm{R}$ be a convex function with $a,b \in J$, $a < b$ then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} . \qquad (6)$$

Proof We will prove the double inequality (6) by expanding conditions (3), (4) and (5) of the Ohlin lemma. Take $a,b \in J$ and $a < b$. Let X, Y, Z be three continuous random variables with measures $\mu_X = \delta_{(a+b)/2}$, μ_Y which is equally distributed in $[a,b]$ and $\mu_Z = \frac{1}{2}(\delta_a + \delta_b$, respectively.

1. We calculate the expectations of the three continuous random variables are X, Y and Z as follow:

$$E(X) = \int_{\mathbb{R}} td\mu_X(t) = \mu_X(\ \{\frac{a+b}{2}\}) \cdot \frac{a+b}{2} \ = \frac{a+b}{2};$$

$$E(Y) = \int_{\mathbb{R}} td\mu_Y(t) = \int_{[a,b]} t \cdot \frac{1}{b-a}dt = (\frac{1}{b-a}) \cdot \frac{a^2-b^2}{2} \ = \frac{a+b}{2};$$

$$E(Z) = \int_{\mathbb{R}} td\mu_Z(t) = a \cdot \mu_z(\{a\}) + b \cdot \mu_Z(\{b\}) = \frac{a+b}{2}.$$

From the calculations, we have shown that the expectations are equal with

$$E(X) = E(Y) = E(Z) = \tfrac{a+b}{2}$$

and we have satisfied condition (3).

2. Observe that the cumulative distributive functions are given by:

$$F_X(t) = \begin{cases} 0, & \text{t } \leq \frac{a+b}{2}, \\ 1, & 1 \ > \frac{a+b}{2} \end{cases},$$

and

$$F_Y(t) = \begin{cases} 0, & \text{t } \leq a, \\ \frac{1}{b-a}t - \frac{a}{b-a}, & \text{t } \in (a,b], \\ 1, & \text{t } > b, \end{cases}$$

and also

$$F_Z(t) = \begin{cases} 0, & t \le a, \\ \frac{1}{2}, & t \in (a, b], \\ 1, & t > b. \end{cases}$$

Using the above calculated cumulative distributions functions $F_X(t)$, $F_Y(t)$ and $F_Z(t)$ we may write the following. For cumulative distributions functions

$F_X(t)$ and $F_Y(t)$ at point $t_0 = \frac{a+b}{2}$ we have

$$t \in (-\infty, \frac{a+b}{2}) \implies F_X(t) \le F_Y(t),$$

and

$$t \in (\frac{a+b}{2}, \infty) \implies F_X(t) \ge F_Y(t).$$

Furthermore for cumulative distributions functions $F_Y(t)$ and $F_Z(t)$ at point $t_0 = \frac{a+b}{2}$ we get

$$t \in (-\infty, \frac{a+b}{2}) \implies F_Y(t) \le F_Z(t)$$

and

$$t \in (\frac{a+b}{2}, \infty) \implies F_Y(t) \ge F_Z(t).$$

From the above observations, we have shown that (4) is satisfied.

3. Since (3) and (4) are fulfilled, we know from the Ohlin lemma that (5) is satisfied. Using (5) we will obtain (6). Observe that for convex orderings $Ef(X) \le Ef(Y) \le Ef(Z)$ we have

$$Ef(X) = \int_{\mathbb{R}} f(t) d\mu_X(t) = \mu_X(\{\frac{a+b}{2}\}) \cdot f(\frac{a+b}{2}) = f(\frac{a+b}{2}) \le$$

$$\le Ef(Y) = \int_{\mathbb{R}} f(t) d\mu_Y(t) = \int_a^b f(t) \frac{1}{b-a} dt = \frac{1}{b-a} \int_a^b f(t) dt \le$$

$$\le Ef(Z) = \int_{\mathbb{R}} f(t) d\mu_Z(t) = \int_a^b f(t) d\mu_Z(t) = f(a) \cdot \mu_z(\{a\}) + f(b) \cdot \mu_Z(\{b\}) =$$

$$= \frac{f(a) + f(b)}{2}.$$

This completes the proof of the double Hermite-Hadamard inequality.

General finite case of Jensen inequality

Theorem 3 is the general finite case of Jensen inequality that is derived from definition 8. This theorem replaces variables $a,b \in J$ with finite variables $x_1,x_2,...,x_n \in J$. We state the theorem and show the proof of it satisfies the conditions (3), (4) and (5) of the Ohlin lemma.

Theorem 3 Let $J \subset$ R be an open interval. If a function f: $J \to$ R is convex, then it satisfies Jensen's functional inequality

$$f(\frac{x_1 + x_2 + ... + x_n}{n}) \leq \frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} \qquad (7)$$

for all $x_1,x_2,...,x_n \in J$, $x_1 < x_2 < ... < x_n$ and $n \in$ N.

Proof We will prove (7) with help of the Ohlin lemma by verifying conditions (3), (4) and (5). Consider $x_1, x_2,..., x_n \in J$ such that $x_1 < x_2 < ... < x_n$. Let X and Y be two continuous random variables with measures $\mu_X = \delta_{(x1+x2+...+xn)/n}$ and $\mu_Y = \frac{1}{n}(\delta_{x_1} + \delta_{x_2} + ... + \delta_{x_n}$, respectively.

1. We calculate the expected values of condition (3) for all $t \in$ R as follows:

$$E(X) = \int_{\mathbb{R}} td\mu_X(t) = \mu_X(\ \{\frac{x_1 + x_2 + ... + x_n}{n}\}) \cdot \frac{x_1 + x_2 + ... + x_n}{n} = $$
$$= \frac{x_1 + x_2 + ... + x_n}{n};$$
$$E(Y) = \int_{\mathbb{R}} td\mu_Y(t) = x_1 \cdot \mu_Y(\{x_1\}) + x_2 \cdot \mu_Y(\{x_2\}) + ... + x_n \cdot \mu_Y(\{x_n\}) =$$
$$= \frac{x_1 + x_2 + ... + x_n}{n}.$$

From the calculations, we have shown that the expected values are equal with

$$E(X) = E(Y) = \frac{x_1+x_2+...+x_n}{n}.$$

2. We want to prove (4), the cumulative distributive functions are given by:

$$F_X(t) = \begin{cases} 0, & \text{t } \leq \frac{x_1+x_2+...+x_n}{n}, \\ 1, & \text{t } > \frac{x_1+x_2+...+x_n}{n}, \end{cases}$$

and

$$F_Y(t) = \begin{cases} 0, & \text{t } \leq x_1, \\ \frac{1}{n}, & \text{t } \in (x_1, x_2], \\ \frac{2}{n}, & \text{t } \in (x_2, x_3], \\ \vdots & \vdots \\ \frac{n-1}{n}, & \text{t } \in (x_{n-1}, x_n], \\ 1, & \text{t } > x_n. \end{cases}$$

Now $F_X(t)$ and $F_Y(t)$ are cumulative distribution functions and for every $t \in \mathbb{R}$ there exists $k \in \{0, 1, ..., n\}$ such that $F_Y(t) = \frac{k}{n}$, we have

$$F_X(t) = 0 \leq \frac{k}{n} = F_Y(t), \text{ for } t < \frac{x_1 + ... + x_n}{n}$$

and

$$F_X(t) = 1 \geq \frac{k}{n} = F_Y(t), \text{ for } t > \frac{x_1 + ... + x_n}{n}.$$

We have satisfied (4) as claimed.

3. Since (3) and (4) are fulfilled above, then we can use (5) to obtain (7). We write

$$f(\frac{x_1 + x_2 + ... + x_n}{n}) = \mu_X(\{\frac{x_1 + x_2 + ... + x_n}{n}\}) \cdot f(\frac{x_1 + x_2 + ... + x_n}{n}) =$$

$$\int_{\mathbb{R}} f(t) d\mu_X(t) = Ef(X) \leq$$

$$\leq Ef(Y) = \int_{\mathbb{R}} f(t) d\mu_Y(t) = \int_{x_1}^{x_2} f(t) d\mu_Y(t) + \int_{x_2}^{x_3} f(t) d\mu_Y(t) +$$

$$... + \int_{x_n-1}^{x_n} f(t) d\mu_Y(t) = f(x_1) \cdot \mu_Y(\{x_1\}) + f(x_2) \cdot \mu_Y(\{x_2\}) + ... +$$

$$+ f(x_n) \cdot \mu_Y(\{x_n\}) = \frac{f(x_1) + f(x_2) + ... + f(x_n)}{n}.$$

Hence we have arrived at (7) and so we have proved that for convex ordering $Ef(X) \leq Ef(Y)$.

All conditions (3), (4) and (5) are fulfilled. Thus, this completes the proof to (7).

Conclusion

In this paper, we observed a fundamental process of establishing elementary knowledge of mathematics, which can be used to derive complex lemmas like the Ohlin lemma. We further observed the application of the Ohlin lemma in the proofs to Theorems 2 and 3 which are the Hermite-Hadamard and the finite case of Jensen functional inequalities. These two theorems are applied to solve problems in areas such as optimization, finance, economics, and operation research where random continuous variables are involved. Having such proved theorems, we can utilize them to model and study the means of the probability distribution functions.

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Author

Cyril Sarsoruo is a lecturer in the Department of Mathematics and Computing Science at DWU. He holds a Master's degree in Theoretical Mathematics and specializes in mathematical analysis. His research interests include functional equations and inequalities. Email: csarsoruo@dwu.ac.pg

Monte Carlo simulations to estimate Pi

Rik King
Peter Anderson

Abstract

A simple Monte Carlo simulation, using functions from various R packages is explored for calculating π using a variety of polygons to circumscribe a unit circle. Starting with a square, higher-order polygons, starting with the hexagon where the ratio of the circle area to the enclosing area to 90.68%, are explored for improved results. From there the number of sides, n, of the circumscribing polygon is progressively increased. The standard error of the estimate is reduced as the number of polygon sides is increased. One of several possible variance reduction methods is discussed.

Keywords: Barycentric Coordinates, Hit or Miss, R, Random variables, Simple Monte Carlo.

Introduction

The value of the transcendental constant π is known to at least 31 trillion decimal places, but even so, interest in finding just the first few digits of the constant using simple processes, has grown over the years; much fostered by internet usage. Figure 1(a) is central to a common Monte Carlo (MC) simulation, where computer-generated random numbers, represented by dots in a square, are classified as to whether they also fall within an enclosed circle. The logic is as follows (Anderson, 2020). If it is assumed that the points are totally random, then:

$$\frac{No\ of\ points\ in\ circle(n)}{No\ of\ points\ in\ square(N)} = \frac{Area\ circle}{Area\ square}. \quad (1)$$

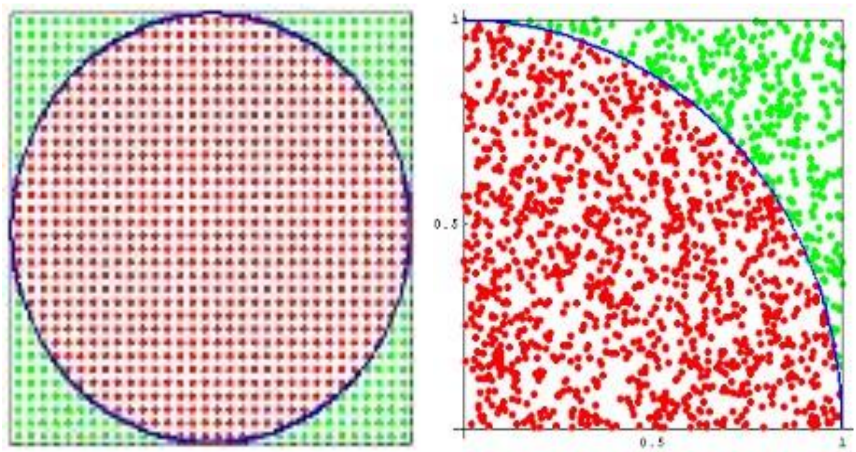
That is

$$\frac{n}{N} = \frac{\pi}{4}. \qquad (2)$$

From the simulation standpoint, the left and right sides of equation (2) provide two different estimators for the value of □, and so, π itself.

The estimators

Figure 1(b) shows a set of X-Y axes set up on the circle/square surface. Any point of the square with coordinates (x, y) will be some distance *d* from the centre where $d = \sqrt{x^2 + y^2}$.



(a) Circle. (b) Quadrant.

Figure 1: Random points.

A point (x, y) lies inside the circle if $d = \sqrt{x^2 + y^2} \leq 1$. Points are either inside or not inside - a “Hit or Miss” (HM) scenario. Simulating *d* is equivalent to simulating a Bernoulli random variable with probability $\frac{\pi}{4}$ ascertained from the right side of equation (2). Thus, naming the estimator as θ_{HM} gives:

$$\widehat{\theta_{HM}} = \sqrt{x^2 + y^2} \leq 1. \qquad (3)$$

$\widehat{\theta_{HM}}$ is an unbiased estimator of $\frac{\pi}{4}$ with Bernoulli variance $N\,\frac{\pi}{4}(1 - \frac{\pi}{4})$ and where *N* the number of trials.

Returning to Figure 1(b) and the RHS of equation (2), the ratio of areas "inside" where random points fall, to the unit area of the quadrant, is just $\frac{\pi}{4}$. Standard calculus can find $\frac{\pi}{4}$ from

$$A = \int_0^1 y\ dx \tag{4}$$

and since for points on the quadrant arc, $x^2 + y^2 = 1$, this becomes[i]

$$\hat{\theta}_I = \int_0^1 \sqrt{(1 - x^2)}\ dx. \tag{5}$$

This simulation estimator $\hat{\theta}_I$ is actually a special case of a general form:

$\hat{\theta}_I = \int_a^b f(x)\ dx,$ with $a = 0$, $b = 1$.

The general integral is approximated by averaging N samples of some function f at uniform random points within an interval. With a set of N uniform random variables $X_i\ \varepsilon\ (a,b)$, and with pdf $\frac{1}{(b-a)}$, the Monte Carlo estimator is:

$$\widehat{\theta_I} = (b - a) \sum_{i=1}^{N} f(X_i).$$

Therefore, there are two different estimators:

(i) $\widehat{\theta_{HM}} = x^2 + y^2 \leq 1$ and (ii) $\hat{\theta}_I = \sqrt{1 - x^2}$.

The variance of $\hat{\theta}_I$ will always be less than the variance of $\widehat{\theta_{HM}}$. Intuitively, this must be so, because (ii) involves the simulation of only one random variable, as opposed to two; and the intuition is easily confirmed algebraically. Interestingly, the latter is derived from the former by the conditioning of x on y. Most of what follows in this article will prefer the estimator $\hat{\theta}_I$.

Alternative circumscribing polygons

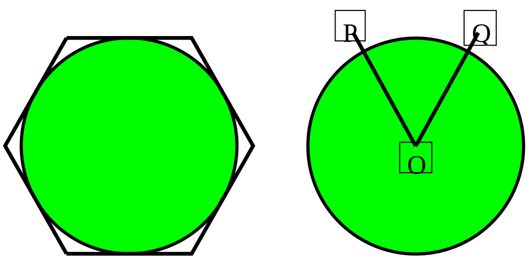
In connection with the circle-in-the-square-problem, not a lot appears to have been written for the cases when the circle becomes circumscribed by some other figure, e.g. a polygon (although the original arithmetic way of Archimedes used just this technique). Is it the case that the spread of the variance of the simulations might be reduced by adopting a different basic arrangement e.g. instead of the circle being enclosed in a square, where it occupies 75.8 percent of the bounding area, would better results be obtained if the circle were enclosed by a higher-order polygon?

Table 1: Circle area as % of Polygon

Polygon	Square	Pentagon	Hexagon	Octagon
% occupied	78.5	86.4	90.68	94.8

Table 1 shows the ratios of some possible figure areas. So, for example, a circumscribed hexagon would bring the ratio of the circle area to the enclosing area to 90.68 percent. This is then a good point to begin a general discussion through the example of a hexagon, moving to the case of more general figures later (it was also a waypoint for Archimedes' arithmetrical/geometrical estimation of the upper limit on the value of π).

Figure 2 shows: (a) a regular hexagon enclosing a circle of unit radius, and (b) one of six triangles comprising the hexagon, and a sector of the circle enclosed. As previously, the centre of the circle is (0,0), with radius 1.



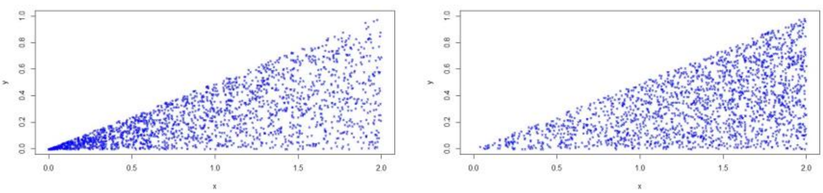
(a) Hexagon, Circle (b) Triangle in Hexagon

Figure 2: Circle and Hexagon

The HM estimator

In the case of the circumscribing hexagon (Figure 2), the triangle shown consists of a tangent and two circle radii extended. Its area includes the circle sector. What is now required is to simulate a set of N random numbers which cover the triangle and ascertain the number n of those numbers also falling within the sector of the circle. Unfortunately, generating random points within a triangular area is more involved than producing random points within a quadrant, which is the appropriate simulation technique for Figure 1(b).

In the case of the quadrant, N random points are generated by R commands such as x = runif (N, 0, 1) and y = runif (N, 0, 1), along the X and Y axes respectively, and uniformly cover the 2D square area. It turns out that an arbitrary triangular area is not covered uniformly by similar commands. Observe $N = 2000$ random points of a triangle in Figure 3 below, one of which was affected by the above method, the other by the correct method (see the programs tri.non uniform.R and tri.uniform.R in the Appendix for the generation of the two triangles)



(a) Non-Uniform. (b) Uniform.

Figure 3: Non-Uniform/Uniform points.

It can be seen that (b) is the way to achieve a pattern of points where there are no obvious gaps in coverage. In fact, (b) relies on a system of barycentric coordinates (Burkhardt, 2014). To use this more complicated procedure is essential since the whole validity of Monte Carlo simulation is posited on random points being uniformly distributed.

Barycentric coordinates

Let the vertices of a general triangle be A, B, C, each with Cartesian vector coordinates (x, y). Let r and s be random numbers where $0 < r < 1$ and $0 < s < 1$. The following intermediate quantities are needed for the coordinates of points P(x, y) lying within the triangle: $ea = 1.0 - \sqrt{s}$, $eb = (1.0 - r) * \sqrt{s}$, $ec = r * \sqrt{s}$.

Then, the barycentric coordinates P are:

$$P = ea * A + eb * B + ec * C. \qquad (6)$$

The inverted triangle (figure 4) has a vertex angle at point O, which is $\frac{\pi}{6}$ radians, so the semi-vertex angle is $\frac{\pi}{12}$. Thus, the length of the tangent side of the triangle is $2\tan(\frac{\pi}{12})$, and the set of triangle(x, y) coordinates to be transformed into barycentric coordinates is therefore:

$$(-\tan(\tfrac{\pi}{12}), 1),\ (0, 0),\ (\tan(\tfrac{\pi}{12}), 1).$$

Fortunately, R incorporates a package 'uniformly' which generates uniform distributions for different geometric shapes. For a triangle, if given the Cartesian vertices' coordinates, it implements the above transformations and generates a vector of N random variates of points. The tabulation below shows estimates of pi and the standard error of estimates for different values of N, from 50 replications of the program polyHM.R (see code in the Appendix).

Table 2: Estimator HM

Hexagon		
N	**pi**	**s.e.**
1000	3.133	0.03
10000	3.142	0.009
100000	3.141	0.003
500000	3.1417	0.0014

But as it is the estimator with the higher variance, the topic of the $\widehat{\theta_{HM}}$ estimator will not be pursued, the other estimator being preferred.

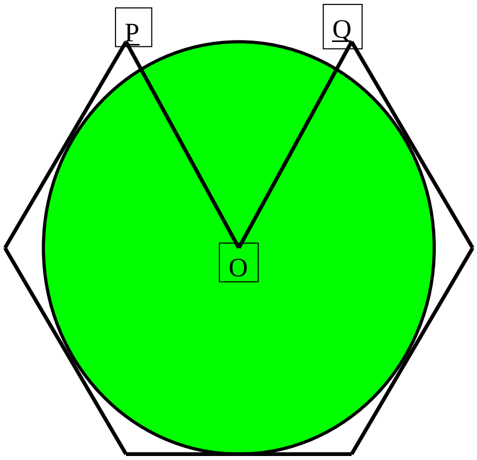


Figure 4: Hexagon and circle

The I estimator

A circumscribing Hexagon (Figures 4 & 5) provides a good illustrative starting point for the I estimator. It shows integration for the sector of the arc RS of $y = \sqrt{1 - x^2}$. The area includes the 2 triangles AOR and BOS; these need to be deducted later for the final result. The advantage is that, from the shape of the figure, barycentric coordinates are not needed.

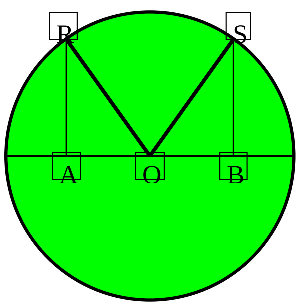


Figure 5: Sector of Hexagon

As with the previous .R program, polyI.R is general, based on the angle of the sector which will always be $\frac{2pi}{n}$, where n is the number of polygon sides. The base angles of the triangles will be (0.5)*(pi - $\frac{2pi}{n}$) and the ± cos of this angle gives the integration limits a and b. The total area of the two triangles (0.5)*sin(1 - $\frac{2pi}{n}$). The program polyI.R employs simulation to estimate the sector area ORS, returning an estimate of *pi*, and the standard error of that estimate. As previously, one argument (n) of polyI(n, N) is the number of sides of the circumscribing polygon.

The following lines of R code will average the results of 50 runs:

```
estims <− replicate (50 , expr = polyI (n, N))
colMeans(t( estims ) )
# n = 4 ,5 ,6 ,8 ,     with N the number of simulations .
```

Table 3: A typical set of results for polyI

Standard Errors				
N	n = 4	n = 5	n = 6	n = 8
1000	0.015	0.010	0.0075	0.0043
10000	0.0048	0.003	0.0023	0.0013
100000	0.0015	0.0010	0.00075	0.00043
500000	0.0006	0.0004	0.00033	0.00019

A typical set of results is shown in Table 3. Clearly, the standard error of the estimate is reduced as the number of polygon sides is increased, which accords with the theoretic probability density considerations (Ebert, 2020).

More decimal places

Even for the best of the results, viz: the octagon with N = 500000, with a standard error of the order of 0.00019, the *pi* estimate is certain only to 3 decimal places. Better is possible.

One variance reduction strategy which is applicable is that of a control variate. A suitable function for the *pi* estimates above is $f(x) = x^2$; the mean value of x^2 in (a, b) is known. It is:

$$\frac{1}{(b-a)} \int_a^b x^2 dx.$$

The program polyIcv.R adds this control variate to polyI.R. Table 4 shows the pi estimates and standard error for 50 runs of polyIcv.R for the case of the Octagon, using the following lines of code:

```
estims <− replicate (50 , expr = polyIcv (8 , N))
colMeans(t( estims ))
```

Table 4: pi estimates and standard error for 50 runs of polyIcv.R

Octagon		
N	*pi*ˆ	s.e.
1000	3.14158	0.000046
10000	3.14159	0.000013
100000	3.1415928	0.000004
500000	3.1415928	0.000002

The last result with standard error of the order of 10^{-6} is quite a satisfactory one, obtained by restricting the number of simulations and there exists the potential for other forms of variance reduction, but that extension will not be continued here.

Summary and conclusion

A simple Monte Carlo simulation, using functions from various R packages, was used to achieve reasonable estimates of the early digits of π using a variety of polygons to circumscribe a unit circle. Polygons were segmented into triangles and barycentric coordinates were used to provide a uniform distribution of points within the triangles from the Monte Carlo simulation tool. The number of polygon sides were increased thereby increasing the ratio of the circle area to the enclosing figure and leading to closer approximations to pi. The standard error of the estimate was reduced as the number of polygon sides increased. One of several possible variance reduction methods was discussed.

References

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Burkhardt. J, (2018). Computational geometry lab. Retrieved 25 August 2020, from https://people.sc.fsu.edu/~jburkardt/classes/cg_2007/cg_lab_monte_carlo_triangles.pdf

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Authors

Rik King (PhD) is a retired Australian academic and now continues his research interests in financial mathematics and simulation methods.

Peter K. Anderson, (PhD) is Head, Department of Mathematics & Computing Science, DWU. Email: panderson@dwu.ac.pg

Appendix

```
# tri . non−uniform
#−−−−−−−−
# non−uniformly distributednumbers over # the triangle (0 ,2) ,
(0 ,0) , (2 ,1)
N <− 2000
x <− 2∗runif (N);
y <− (x/2)∗ runif (N);
rnums <− matrix(c(x , y) , nrow = N);
plot .new() plot(rnums , xlim = c(0 ,2) , ylim = c(0 ,1) , pch = '∗
' , col = "blue" , xlab = 'x ' , ylab = 'y ' )
#−−−−
# tri . uniform
#−−−−
# uniformly distributed numbers over # the triangle (0,2), (0,0),
(2,1)

library ( uniformly )
N <− 2000 tri . uniform <− runif in triangle (N, c(0,0) ,c(2,0)
,c(2,1))
plot .new() plot( tri . uniform , xlim = c(0,2) , ylim = c(0,1) ,
pch = '∗ ' , col = "blue" , xlab = 'x ' , ylab ='y ' )

#−−−−
# polyHM
#−−−−

library ( uniformly )
polyHM <− function (n, N){ g <− function(u, v){uˆ2 + vˆ2}
tp <− tan( pi/n)
rit <− runif _in_triangle (N, c(−tp ,1) , c(0 ,0) ,c(tp ,1))
x <− rit [ ,1]; y <− rit [ ,2]; p <− g(x , y)
inside <− p[p <= 1]; num <−length( inside )
pd <− (n∗tan( pi/n )); Ex _ val <− (num/N);
pi_est <− Ex_val∗pd; se <− pd∗sd( inside )/sqrt(N)
```

```
return(c( piest , se )) } # end function
# example
estims <− replicate (50, expr= polyHM(6,1000))
colMeans(t( estims ) )

#−−−−−−−−
# polyI
#−−−−−−−−
polyI <− function(n, N){
 g <− function(u){sqrt(1 − uˆ2)}
 x <− cos ((0.5)∗( pi − 2∗pi/n))
 b <− x ; a <− − x ; Y <− runif (N, a , b)
 tri . area <− 0.5∗sin ((n − 2)∗pi/n)
 estim <− n∗((g(Y))∗(b−a) − tri . area )
 me <− mean( estim ); se <− sd( estim )/sqrt(N)
return(c(me, se ))} # end function
# example estims <− replicate (50 , expr = polyI (6 , 500000))
 colMeans(t( estims ) )
#−−−−−−−
# polyIcv
#−−−−−−−
polyIcv <− function(n,N){ options( digits = 8)
g <− function(u){sqrt(1 − uˆ2)} x <− cos ((0.5)∗( pi − 2∗pi/n))
 a <− − x ; b <− x ; Y <− runif (N, a , b );

#− control variate Z−
 Z <− Yˆ2;
 cv <− function(w){wˆ3/3} # the mean for xˆ2
 Zm <− (1/(b − a))∗(cv(b) − cv(a) )
#−−−−−−−
# simulated mean of Y Yms <− mean(Y); tri . area <− 0.5∗sin
((n − 2)∗pi/n)
#−−−−−−−
 X <− g(Y); Xms <− mean(X)
 c <− −sum ((X − Xms)∗(Z − Zm))/sum((Z − Zm)ˆ2)
```

```
tmp <− (b −a)∗(X + c ∗(Z − Zm))
estim <− n∗(tmp − tri . area )
me <− mean( estim ); se <− sd( estim )/sqrt(N)
return(c(me, se )) }# end function
# example estims <− replicate (50 , expr = polyIcv (8 ,
 500000)) colMeans(t( estims ) )
#
```
