Some interesting properties of mathematical entities to challenge and motivate students

Ram Bilas Misra
Peter K Anderson

Abstract

The numbers have ever attracted the best brains round the world since time immemorial and still present unsolved problems which are capable of challenging keen young minds to embrace and continue their study of mathematics. Certain special characteristics of integers and factorials are discussed as well as various paradoxes which have puzzled mathematicians down the ages.

Key words: numbers, integers, factorials, paradoxes, unsolved problems.

Introduction

Approximately 65% of Papua New Guinea's current GDP derives from mineral resources which will not last forever. Eventually, availability of natural resources will diminish and the country's survival will depend on how well it develops its human resource. Inadequate emphasis on mathematics and computer science education will eventually be reflected in weak industrial growth. Mathematics has contributed to the revolution of Information Technology. We note also that computer science has played a vital role in the development of areas such as the information technology (IT) industry, manufacturing industry, satellites, e-banking, e-commerce, telecommunications, the global positioning system (GPS), geographic information system, and remote sensing. The Government needs to allocate adequate resources for basic science and engineering education at all levels if PNG is to gain recognition as a developed nation. It may be interesting to briefly touch upon some historical aspects of the core subject: especially Mathematics.

This paper will focus on aspects of mathematics education and its motivation. We note that ancient mathematics began with attempts to analyse space and time. The concept of number was one of the earliest tools used, firstly just as the whole or counting numbers. When negative numbers were added, we had the integers. Then, followed the rational numbers (fractions) and the irrational numbers (unable to be expressed as finite decimals) - the latter two types constituting the Real Numbers. The study of the real numbers, known as real analysis constitutes much of classical (as distinct from modern) mathematics and what our undergraduate mathematics students study even to this day. In all of these number systems, mathematicians have long since been fascinated by the apparently fortuitous and interesting patterns which can be discovered. High speed computers encourage the search for more of these properties.

This paper seeks to display certain interesting properties of the numbers as an introduction to topics which may motivate our students to develop a fascination for even elementary mathematics. It firstly considers some least positive integers expressible in two different ways firstly as the sum of distinct positive integers, and then as squares of distinct positive integers, as cubes of distinct positive integers, and finally as fourth powers of distinct positive integers. It will then be noted that finding similar expressions for higher powers of distinct positive integers is a still unsolved or perhaps unsolvable problem. Other proper- ties of numbers involving factorials and integral powers of their digits will also be discussed. Finally, some more facts, myths and paradoxes will be included. We first consider integers.

Interesting properties of integers

Distinct positive integers

Integers are the positive and negative whole numbers including zero. The least integer expressible as the *sum* **of** distinct positive integers in two different ways is 5:

$$1 + 4 \ = \ 5 \ \ (\textit{least}) \ = \ 2 + 3.$$

As a special case, if repeating numbers are included, 4 is such least integer:

$$1 + 3 \ = \ 4 \ \ (\textit{least}) \ = \ 2 + 2.$$

Sums of squares

The least integer expressible as the *sum of squares* of distinct positive integers in two different ways is 125. Thus we have:

$$(2)^2 + (11)^2 \ = \ 125 \ \ (\textit{least}) \ = \ (5)^2 + (10)^2.$$

As a special case, if repeating numbers are included, 50 is the least integer:

$$(1)^2 + (7)^2 \ = \ 50 \ \ (\textit{least}) \ = \ (5)^2 + (5)^2.$$

Sums of cubes

The least integer expressible as the *sum of cubes* of distinct positive integers in two different ways is found to be 1729. Thus we have:

$$(1)^3 + (12)^3 \ = \ 1729 \ \ (\textit{least}) \ = \ (9)^3 + (10)^3.$$

It may be noted that there is no smaller number which could be so expressed even for repeating numbers.

We note here a fascinating story about this number discovered by Indian prodigy (Srinivas Ramanujan, FRS) born on 22.12.1887 at Erode (near Chennai) in present day's Tamilnadu State of India (Puiu, 2015). He left this mortal world on 26.4.1920 because of TB. He was offered a Research Fellowship at the Trinity College, Cambridge (U.K.) and was working with Prof. G.H. Hardy. Having contracted TB, he was hospitalized in Cambridge. One day (in 1918) Hardy visited the Hospital to console the dying young mathematician. By then Hardy had come to realize that discussion on numbers alone could console Ramanujan. So, he mentioned the registration number

(1729) of the taxi by which he had travelled. Ramanujan immediately remarked that the number is of special significance. It is the least number expressible in two different ways as the sum of distinct integers as demonstrated above.

Sums of fourth powers

After that extraordinary insight, Hardy decided to ask Ramanujan about the least number possessing the same property for the *fourth* powers of some numbers. Both of these two giant mathematicians of their era were unaware about the discovery made by another mathematical prodigy (Leonhard Euler) around 148 years before this historical dialogue was going on between them.
The Hungarian born (American Professor) Paul R. Halmos quoted in his book 'Naïve Set Theory' (Halmos, 1960) that:

$$(133)^4 + (134)^4 = 635{,}318{,}657 \ (\textit{least number}) = (158)^4 + (59)^4.$$

A further set of challenging integer problems can now be considered.

Sums of fifth powers

We seek to find the least numbers possessing the similar characteristics for higher powers 5, 6, 7, etc. We seek to explore the solutions of equations given by:

$$(a)^n + (b)^n = \textit{least number} = (c)^n + (d)^n,$$

for integers a, b, c, d where n could be 5, 6, 7, etc.

It is disappointing to note that such problems are still unsolved. Some computer experts at Eritrea Institute of Technology, Asmara (Eritrea), North-East Africa made attempts in 2011 to tackle the problem for $n = 5$. The computer stopped after having run for a whole night without discovering the desired solution.

A second area of interest for this paper is certain properties of factorials.

Interesting numbers involving factorials

Factorials

Factorials are defined as the continued product of a number (usually a positive integer), say n, with its recurring lower numbers continually diminished by 1, i.e. $n - 1, n - 2, \ldots, 1$. It is denoted by the symbol $n!$. The factorial of 0 (zero) has been defined as 1. We now consider some interesting properties and unsolved problems.

Sums of factorials

The following integers are expressible as the sum of factorials of their own digits:

$$1 = 1!, \qquad 2 = 2!, \quad 145 = 1! + 4! + 5!,$$

and

$$40585 = 4! + 0! + 5! + 8! + 5!$$

Finding the next number possessing similar characteristics is still an unsolved problem (Misra, 2010).

Numbers involving powers of their digits

The following integers are expressible as the sum of powers of their own digits raised by themselves:

$$1 = (1)^1, \qquad 3435 = (3)^3 + (4)^4 + (3)^3 + (5)^5$$

$$= 27 + 256 + 27 + 3125.$$

Again, finding the next number possessing such characteristics remains an unsolved problem. (Iyengar, 1967).

Some mathematical problems remain of permanent interest over the history of time. Of late, British schools have started teaching some principles of "*Vedic Mathematics*" (Vedic Mathematics Academy) which were propounded by ancient Indian scholars many thousand years back. The oldest book ever written, the *Rigveda*, deals with 16 computing formulae, so-called "*Sulba Sutras*". The first of these called "*Ekadhiken Poorven*" magnificently describes the simplest way to obtain an accurate value of the fraction 1/19 (up to 18 decimal places) without carrying out an actual division process:

$$1 / 19 = \cdot 0526\ 3157\ 8947\ 3684\ 21.$$

We begin with the numeral 1 (the digit in the 10s column of the number 19), double it to precede 1, and then continue this process of doubling each (single) digit so obtained and preceding it the digit doubled. We repeat the process 18 times to get above result. Remarkably enough, the above "*Sutra*" (the formula) was given by an intellectual brain. Surprisingly, it gained the attention of the scholars of the Western world so late.

Prime numbers

An integer m with only factors $\pm$ 1 and $\pm$ m (i.e. itself and 1) is called a *prime number*. Thus the first prime number is 2. The largest known so far prime number was discovered by a German ophthalmologist Dr. Martin Nowak on 18th February, 2005 (Gruener, 2005).

It is

$$2^{25964951} - 1$$

having 7,816,230 digits.

Some more facts, myths and paradoxes

It is of course well known that two plus three always adds up to five. However, it is only so when the nature of the objects being counted is the same. Two boys plus three girls makes neither five boys nor five girls.

Modern mathematics

Today's mathematics, so-called *modern mathematics*, is far moved from simply working with numbers. Rather it is based on the set-theoretic notions bearing a close proximity with logic and philosophy. As an example of a paradox, we take into account the fact that the rivers "the *Ganges*" and "*Yamuna*" converge into a single river at Allahabad (India) after which, it is known as the *Ganges* alone. Interpreting this mathematically, we note that:

The Ganges + Yamuna = the Ganges.

It may also be noted that the operation of addition or literally speaking *mixing* of two distinct substances is not the same as the
ordinary addition of two numbers. Thus we have:

One kilogram of sugar + one litre of water = One litre of drink.

Development of the Theory of Relativity

The Theory of Relativity as proposed by Nobel Laureate Professor Albert Einstein was arguably conceived much earlier by an ancient scholar St. *Tulsdidas* (1497-1623 A.D.), a celebrated Hindi poet and philosopher. He wrote the Hindu epic *Ram-Charit-Manas* popularly called *Ramayana* (history of Rama) and was probably the one to have put it into writing. As with Einstein, it was argued that when the velocity of a moving body approaches the velocity of light then the length of the body contracts to zero but its mass tends to infinity. Likewise, describing the sudden jump of *Hanuman* into the mouth of devil *Sursa* with exorbitant velocity, Tulsidas concludes the tiny figure of Hanuman:

"*Ati laghu roop Pawan-sut keenha* !"

(i.e. Hanuman's body appeared very small).

Some mathematical paradoxes

Finally, some *paradoxes* both mathematical and linguistic are now presented. A statement free from self-contradictions and giving a definite meaning, whether in affirmation or in negation, is called a *mathematical statement*. For instance, the claim that "We are writing a paper on mathematics" becomes a mathematical statement if it is either true or false. In the case of an indefinite situation (neither true nor false), a statement cannot be accepted as mathematical. Such claims are called mathematical *paradoxes*. A celebrated example of a paradox is presented in the following statement:

"A *barber claims to shave* (*all and only*) *those who do not shave themselves*".

We then ask the question: Who shaves the barber? The claim is not a mathematical statement as it fails to conclude any definite answer to this question. Thus, it cannot be classified as either true or false. Beginning with the (probable) answer that he shaves himself, it can be shown to contradict the claim. On the other hand, if he does not shave himself, he must be included amongst those to be shaved by him. It establishes that he has to shave himself which is again in contradiction to the initial claim. Thus, either way we are unable to derive any definite conclusion.

Similarly, the statement "*A liar claims that he never tells a lie*" cannot be described as either true or false (if true it is false, if false true) and so is not a mathematical statement.

Such paradoxes were found at the foundations of mathematics as 20th century mathematicians sought to establish all mathematics in a solid logical structure similar to that found in Euclid's geometry. The paradoxes undermined their efforts to place all of mathematics on such a stable foundation. We could also note in passing the weakness in Euclid's geometry built on foundational definitions of points (dimensionless), lines (1 dimension) and planes (2 dimensions), yet all objects lying in the real world having 3 dimensions.

Summary and conclusions

This paper has sought to provide a short introduction to many interesting properties of familiar number systems, including integers and factorials which could be used to challenge and motivate students of mathematics. It also includes unsolved problems indicating that mathematics is not a closed body of knowledge but like the physical sciences is also involved in making new discoveries with the human mind rather than empirical measurements. It also mentions some paradoxes which probe the very foundations of mathematics.

Acknowledgements

Authors wish to express their gratitude to the Divine Word University in general to have very kindly provided opportunities to them to prepare the article. The first author also wishes to record ever inspiring and constant encouragement extended by Profesor Peter K. Anderson.

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Authors' details

Professor Ram-Bilas Misra, PhD
Department of Mathematics & Computing Science
Divine Word University, PNG
Email: rmisra@dwu.ac.pg

Professor Peter K Anderson, PhD
Foundation Head, Department of Information Systems, and Head, Department of Mathematics & Computing Science
Divine Word University, PNG
Email: panderson@dwu.ac.pg