Some unique characteristics of exponential functions

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Abstract

The invariance of the exponential function under successive levels of differentiation is explored. Invariant functions for up to 5 levels of differentiation are determined. Some examples which leverage this invariance property are discussed. By contrast, the Gaussian function *exp* $(-x^2)$ is shown to be diverse when compared with the original function rather than under invariant under similar differentiation. Similarly, results for the moment generating function for the normal distribution are diverse except under very limited conditions.

Keywords: Exponential functions, ordinary differential equations, invariance, and differentiation

Introduction

The exponential function e^x is a unique function because the derivative of the function is the same as the function or alternately we can state that it is invariant under differentiation, in fact, under repeated differentiations. Having this property makes it both unique among the known functions and also one of the most important functions in pure and applied mathematics.

This paper will derive the invariant property and illustrate its use in modelling, growth and decay of physical and biological situations to motivate further exploration of invariant functions with respect to their variables up to six levels of differentiation. At the second level, an application of the Ordinary Differential Equation (ODE) will be made to model vibrating systems to motivate exploration of further levels. Similar invariance or otherwise will also be investigated for related functions: e^{-x} and e^{-x^2} .

Unique property of the exponential function e^x

The exponential function e^x is a unique function because the derivative of the function is the same as the function. We consider $f(x)$ to be a function such that:

$$d\,f(x)/dx = f(x),$$

which can be alternatively presented (treating the derivative as a differential and cross multiplying) as:

$$d\,f(x)\,/\,f(x) \;=\; dx.$$

Integrating both sides with respect to their respective variables, we have:

$$\ln f = x, \qquad (1)$$

which can be written equivalently as:

$$f(x) = e^x.$$

Allowing for an arbitrary constant, c, in the right hand side of (1) after integration, we obtain a general function:

$$e^{x+c} = e^x . e^c = b.e^x,$$

where $b = e^c$ is a constant.

We can also notice that it follows that the derivatives of any order of the function e^x will also remain the same. Thus we can write:

$$D^n e^x \equiv d^n e^x/dx^n = e^x, \qquad (2)$$

where D is treated as an operator d/dx and n is a whole number.

Alternate derivation

This result can also be obtained by considering e^x as defined by the power series expansion:

$$e^x = \sum_{n=\infty}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

or, alternately by:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n .$$

Then, by taking derivatives of both sides of the equation, we have:

$$\frac{d\ e^x}{dx} = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots.\right) = \left(0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} \ldots.\right)$$

$$= 1 + x + \frac{x^2}{2!} \ldots\ldots\ldots = e^x,$$

thereby, confirming (2) for $n = 1$, since this process can be repeated n times always with the same result. Thus, we can conclude that the function e^x remains invariant under differentiation of any order.

Significance of uniqueness property

Exponential e^x is one of the most important functions in pure and applied mathematics because of this very important uniqueness property:

$$d\, e^x / dx = e^x,$$

which can be interpreted as the rate of growth at any point (e.g. in time) of the function being equal to the value of the function at the point. Thus, it can be used in first order linear differential equations [5] to model many bio-logical and other systems where individual or population growth (or decay) rate depends on the present state (e.g. size) of the system.

We also note that $y = e^x$ is the solution to the differential equation $dy/dx = y$. Further, if $f(x) = c.\, e^{kx}$ where c and k are arbitrary constants then:

$$f'(x) = \frac{d(ce^{kx})}{d(kx)} \cdot \frac{d\ \ kx}{dx} = k\, c\, e^{kx} = k\, f(x).$$

Here, c is an arbitrary constant making $f(x) = c.e^{kx}$ a general solution to the ODE $d f/dx = f$.

Because of the widespread usefulness of this exponential function, we now explore the behaviour of some related functions.

Function with invariant 2nd order derivatives

We now seek to demonstrate similar properties of the function, e^{-x}, the reciprocal of e^x, as the only other function whose derivatives and integrals are the same. We consider $f(x)$ to be a function such that:

$$d f/dx = \int f(x) dx,$$

which can also be presented, after taking the derivative of both sides, as:

$$D^2 f = d^2 f/dx^2 = f(x), \qquad \text{or,} \qquad (D^2 - 1) f(x) = 0. \tag{3}$$

Here, we have a linear second order Ordinary Differential Equation (ODE). Results obtained so far suggest we try a solution of the form $e^{\lambda x}$, where λ is a parameter. The equation (3) then becomes:

$$(\lambda^2 - 1)\, e^{\lambda x} = 0.$$

Since $e^{\lambda x} \neq 0$, we have an auxiliary or characteristic equation:

$$(\lambda^2 - 1) = 0,$$

which has two real roots: $\lambda = \pm 1$. Hence, the most general solution of ODE (3) can be written as:

$$f(x) = C_1\, e^x + C_2\, e^{-x}, \tag{4}$$

for e^x and e^{-x} being linearly independent functions, and C_1 and C_2 are arbitrary constants to be determined by initial conditions. Considering $C_1 = 1$ and $C_2 = 0$, as special cases, we obtain the solution e^x; and for $C_1 = 0$ and $C_2 = 1$, the second solution e^{-x}.

Thus, the ODE (3) for the function e^{-x} becomes an alternate version of the ODE (2) for the function e^{-x}, when $n = 2$.

We can now claim that the exponential function e^x and its reciprocal e^{-x} are the only functions whose second order derivatives remain the same.

Further functions involving e^x

We now consider special functions obtained from (4) by setting values of the constants $C_1 = \pm C_2$ to 1/2. These values define the hyperbolic (trigonometric) functions:

$$\cosh x \equiv (e^x + e^{-x})/2, \qquad \sinh x \equiv (e^x - e^{-x})/2.$$

We can therefore conclude that the second order derivatives of the hyperbolic functions cosh x, sinh x and their linear combinations:

$$f(x) = C_1.\cosh x + C_2.\sinh x, \qquad (5)$$

are also invariant under differentiation.

Application to second order linear ODE

Vibration of damped mechanical systems can be modelled as a second order linear ODE with equation given by:

$$m\ddot{x} + c\dot{x} + kx = 0, \qquad (6)$$

which represents the set of forces involved in a transient vibration [5]. Here $\ddot{x}$ is acceleration (second derivative of position with respect to time), $\dot{x}$ is velocity (first derivative of position with respect to time), and c and k are constants of damping and elasticity respectively. Equation (6) can also be written in the form:

$$\ddot{x} + k\dot{x} + \omega^2 x = 0, \qquad (7)$$

where $k = c/m$ and $\omega^2 = k/m$, where ω is the angular velocity.

Here we seek a function which is invariant under the sum of both second and first order derivatives. Results obtained so far suggest we try functions of the form $x = e^{\lambda t}$, which on substitution in (7), yield the following characteristic equation:

$$\lambda^2 + k\lambda + \omega^2 = 0. \qquad (8)$$

This is a quadratic equation in λ with solutions:

$$\lambda = -k/2 \pm \sqrt{(k^2/4 - \omega^2)}. \qquad (9)$$

There arise three solutions: two real and one complex, leading to three physical situations (over damping, critical, and under damping shown in Figures 1, 2 and 3 respectively) depending on the possible values of the discriminant, $k^2/4 - \omega^2$, as follows:

(i) $k^2/4 > \omega^2$ with two real roots $x = c_1\, e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$,

(ii) $k^2/4 = \omega^2$ a real double root $x = (c_1 + c_2 t)e^{-kt/2}$,

(iii) $k^2/4 < \omega^2$ two complex roots $x = e^{-kt/2}(A \cos \omega t + B \sin \omega t)$

leading to decaying vibratory motion (Figure 3).

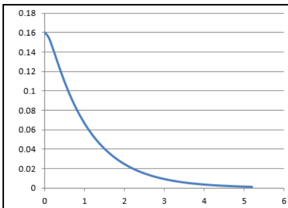

Figure 1. Over damping from 2 real roots λ_1 and λ_2. Damping occurs so rapidly that oscillation does not occur [1], p. 68).

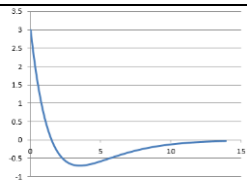

Figure 2. Critical damping from 2 equal roots [1], Solution p. 56.

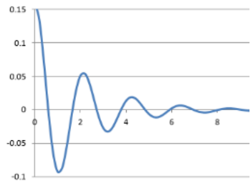

Figure 3. Under damping from complex conjugate roots [1], p. 68.

Functions with invariant third order derivatives

Here we consider the ODE (2), for $n = 3$, and seek a function $f(x)$, again other than e^x satisfying the equation:

$$D^3 f = f(x),$$

which can be written as:

$$(D^3 - 1) f(x) = 0, \qquad (10)$$

with auxiliary equation $\lambda^3 - 1 = 0$. This equation has one real root, 1, and two complex roots: $\lambda = -1/2 \pm (\sqrt{3}/2)\, i$.

Hence, the most general solution of the ODE (10) will be [2], p. 14

$$f(x) = C_1.e^x + C_2.e^{-x/2}.\cos\{(\sqrt{3}/2)\, x + C_3\}, \qquad (11)$$

where C_1 and C_2 are arbitrary constants as previously. Thus, we have that the most general functions. with third order derivatives invariant, represented by equation (11). Choosing particular values of the constants:

(i) $C_1 = 1$ and $C_2 = 0$, we obtain the special function e^x,

(ii) $C_1 = 0$, $C_2 = 1$ and $C_3 = 0$, we obtain another special function:

$$e^{-x/2}.\cos(x\sqrt{3}/2),$$

which remain invariant under third order differentiation. It can also be noted that all even order derivatives of the function e^{-x} also remain invariant.

Thus, we can claim that the most general functions given by equations (4) and (5) also possess invariant derivatives of any even order:

$$D^{2n} f(x) = f(x), \qquad (12)$$

where n is a whole number. Clearly, the special functions e^x and its reciprocal ones also satisfy equation (12).

Functions with invariant fourth order derivatives

As seen from equation (12), the functions e^x and its reciprocal possess the same fourth order derivatives. Similar, is the case with their linear combinations given by equation (4):

$$D^4 f(x) = f(x), \qquad (13)$$

where $f(x)$ is given by equation (4).

It may be easily verified that the exponential functions e^{ix} and e^{-ix} also possess the same 4th order derivatives. Accordingly, the 4th order derivatives of circular (trigonometric) functions:

$$\cos x \equiv (e^{ix} + e^{-ix})/2, \qquad \sin x \equiv (e^{ix} - e^{-ix})/2i,$$

also remain invariant. Thus, we can claim that the 4th order ODE (13) also holds for $f(x) = \cos x$, or $\sin x$, or their linear combinations:

$$f(x) = C_{1.} \cos x + C_{2.} \sin x.$$

Functions with invariant fifth order derivatives

Here we seek any functions $f(x)$ other than e^x whose fifth order derivatives are invariant satisfying the ODE:

$$D^5 f = f(x), \qquad \text{or} \qquad (D^5 - 1) f(x) = 0. \qquad (14)$$

Here the auxiliary equation $\lambda^5 - 1 = 0$ has only one real root 1, and four complex roots. Solving this 5th degree algebraic equation using *De Moivre's* theorem from trigonometry, we obtain the roots:

$$\lambda^5 = 1 \quad = \cos 2n\pi + i \sin 2n\pi = e^{2n\pi i},$$

and so, taking the 5th root of each side, we have $\lambda = e^{2n\pi i/5}$, where $n = 0, 1, 2, 3, 4$. These values of n determine the corresponding values of λ:

$$\lambda_1 = 1, \quad \lambda_2 = e^{2\pi i/5}, \qquad \lambda_3 = e^{4\pi i/5} = -e^{-\pi i/5},$$

$$\lambda_4 = e^{6\pi i/5} = -\,e^{\pi i/5}, \qquad \lambda_5 = e^{8\pi i/5} = e^{-2\pi i/5}.$$

Hence, we can state the most general solution of ODE (14) as:

$$f(x) = C_1.e^x + C_2.e^{m_2.x} + C_3.e^{m_3.x} + C_4.e^{m_4.x} + C_5.e^{m_5.x}, \qquad (15)$$

where C's again are arbitrary constants and m_n replace λ_n. Assigning particular values to C's, one may also derive e^x and other functions as special cases.

Further exploratory work would be required to explore the functions other than e^x which may possess invariant derivatives of odd orders greater than 5.

Another interesting function, we can similarly investigate, is the well-known Gaussian function which is used to plot frequency distributions of populations in inferential statistics and which will now be considered.

Gaussian function $f(x) = e^{-x^2}$

Gaussian function e^{-x^2} (Figure 4) is used in probability theory and statistics to model the normal frequency distribution [4]. Computing a sequence of derivatives of e^{-x^2} with respect to x, we have respectively:

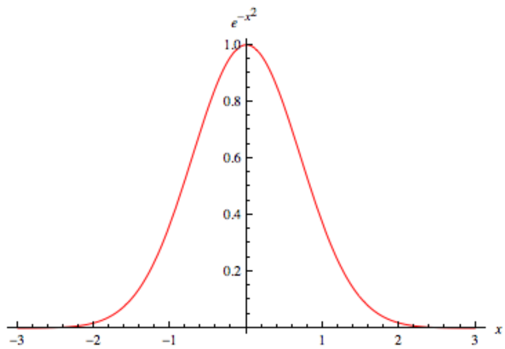

Figure 4. Gaussian function e^{-x^2} used to model frequency distributions.

$$D f = d f/dx = -\,2x.\,f(x), \qquad D^2 f = (-\,2 + 4x^2).\,f(x),$$

$$D^3 f = (12x - 8x^3).\,f(x), \quad D^4 f = (12 - 48x^2 + 16x^4).\,f(x),$$

we can note their diverse rather than invariant character under successive derivation, when compared to the original Gaussian function $f(x)$. However, it is interesting to note the equivalence of the function and its first derivative at the point $x = -1/2$ where:

$$f(-1/2) = [D f]_{x=-1/2} = e^{-1/4} = 1/e^{1/4}.$$

Its successive derivatives at above point are in linear relationship, i.e. proportional to $f\,(-1/2)$ as shown in the following:

$$D^2 f = -e^{-1/4}, \quad D^3 f = -5e^{-1/4}, \quad D^4 f = e^{-1/4}, \qquad \text{etc.}$$

Moment generating function of normal distribution

The moment generating function (MGF) $f\,(x)$, for the normal distribution can be written in the simplified form [3], Lesson 9:

$$f\,(x) = \exp\,(C_1.\, x + C_2.\, x^2), \tag{16}$$

where $C_1 = \mu$ and $C_2 = \sigma^2/2$ are the mean and half the variance of the distribution respectively. Factorizing, the product function (16) can be written as:

$$f\,(x) = \exp\,(C_1.\, x).\, \exp\,(C_2.\, x^2).$$

For the special cases $C_1 = 0$ and $C_2 = -1$, $f\,(x)$ reduces to the previously discussed Gaussian function e^{-x^2} . Further, using $C_1 = \pm 1$ and $C_2 = 0$ we have the exponential functions also previously discussed. Computing the successive derivatives of the function, we have:

$$D\,f\,(x) = (C_1 + 2C_2.\, x).\, f\,(x),$$

$$D^2 f\,(x) = \{2C_2 + (C_1 + 2C_2.\, x)^2\}.\, f\,(x),$$

$$D^3 f\,(x) = (C_1 + 2C_2.x).\{6C_2 + (C_1 + 2C_2.x)^2\}.f\,(x), \text{ etc.}$$

Thus, for general values of the constants C_1, C_2 there is no linear relationships between these derivatives with the function unless C_2 is zero. In that case, as seen above, the MGF $f\,(x)$ reduces to a general exponential function $\exp\,(C_1.\, x)$ only.

Considering the natural logarithm of the MGF $f\,(x)$, we have:

$$\ln f = C_1.\, x + C_2.\, x^2,$$

which yields:

$$D \ln f = C_1 + 2C_2.\, x, \qquad D^2 \ln f = 2\, C_2,$$

with further order derivatives being zero for C_2 being constant.

Summary and conclusions

Five levels of repeated differentiation have been explored and the invariant functions determined (Table 1) showing that they all include the exponential functions as components.

Table 1. Invariant functions for 5 levels of differentiation have been determined.

ODE	Invariant Functions
$(D^n) f(x) = f(x)$	$f(x) = e^x$, invariant for all whole numbers n.
$(D^2 - 1) f(x) = f(x)$	$f(x) = C_1 e^x + C_2 e^{-x}$, C_1 and C_2 constants.
$(D^3 - 1) f(x) = f(x)$	$f(x) = C_1 e^x + C_2.e^{-x/2}.\cos\{(\sqrt{3}/2)x + C_3\}$,
$(D^4 - 1) f(x) = f(x)$	$f(x) = (e^{ix} + e^{-ix})/2$, $f(x) = (e^{ix} - e^{-ix})/2i$, and $f(x) = C_1 \cos x + C_2 \sin x$
$(D^5 - 1) f(x) = f(x)$	$f(x) = C_1.e^x + C_2.e^{m_2 .x} + C_3.e^{m_3 .x} + C_4.e^{m_4 .x} + C_5.e^{m_5 .x}$

In general, it has been shown that invariant functions for lower levels of differentiation can be obtained as special cases of the functions invariant under higher levels of differentiation. By contrast, the Gaussian function e^{-x^2} was shown to be diverse when compared with the original function rather than under invariant under successive differentiation. Similarly, results for the moment generating function for the normal distribution were shown to be diverse except under very limited conditions.

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